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SELF-SIMILARITY, L^p-SPECTRUM AND MULTIFRACTAL FORMALISM

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ABSTRACT. This is an expository survey of recent work on self-similar measures centered around the L^p-spectrum and its relationship to the local dimension spectrum. The relationship is the multifractal formalism proposed by physicists. We will treat the formalism rigorously here. The open set condition and a new weaker separation condition will be discussed in detail; several techniques for calculating the L^p-spectrum will be introduced; and the multifractal structure of functions satisfying the two-scale dilation equations will also be discussed.

§1. Introduction. For a finite family of contractive maps $\{S_j\}_{j=1}^m$ on $\mathbb{R}^d$, there exists a unique compact subset K in $\mathbb{R}^d$ satisfying $K = \bigcup_j S_j(K)$. The set K can be obtained by iterating the maps through the *cascade algorithm*, starting from any fixed bounded set or point. For this reason we call $\{S_j\}_{j=1}^m$ an *iterated function system* (IFS) and K the *attractor* of the system. If we associate a set of probability weights a_j to each of the S_j, then the iteration will produce a unique probability measure μ satisfying

$$\mu = \sum_{j=1}^{m} a_j \mu \circ S_j^{-1}. \tag{1.1}$$

In particular when the S_j's are contractive similitudes, i.e., $S_j x = \rho_j R_j x + b_j$, $0 < \rho_j < 1$, R_j a linear isometry and $b_j \in \mathbb{R}^d$, we call the above K a *self-similar set* and μ a *self-similar measure.* If in addition the $S_j K$'s are disjoint, then each $S_j K$ will be an identical copy of K, the same holds for the μ restricted to $S_j K$.

This basic concept of self-similarity was introduced by Mandelbrot in his momentous monograph [M1], the above mathematical set-up was given by Hutchinson in [Hut] and the iterated function system notion was invented by Barnsley [Ba]. Because of its simplicity and fundamental nature, self-similarity has been playing a central role in fractal geometry and has many ramifications,

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e.g., self-affinity, self-conformality, non-map specifying (Moran) self-similarity and statistical self-similarity.

There are two interesting recent developments on self-similarity concerning the L^q-spectrum. Strichartz raised the following general questions: Whereas Lebesgue measure is self-similar and is the most fundamental entity in harmonic analysis on Euclidean space, what happens if we replace Lebesgue measure by other self-similar measures? How does a fractal or a self-similar object express itself in the Fourier transformation side? In a series of papers ([Str1-6, STZ, JRS]), Strichartz *et al* considered a wide range of topics including the fractal Plancherel theorems, Fourier asymptotics and convolution equations of self-similar measures, self-similar tilings, and extensions to stratified nilpotent Lie groups. These results are contained in his very informative survey paper [Str6]. In this development one of the most important concepts is the L^q-dimension of a measure defined by

$$\dim_q(\mu) = \underline{\lim}_{h\to 0^+} \frac{\ln \sum_i \mu(Q_i)^q}{(q-1)\ln h} \qquad q > 1,$$

where $\{Q_i\}_i$ denote the family of h-mesh cubes. It was first defined by Renyi [Re2, Chapter 9] as a natural generalization of the entropy dimension and various techniques have been developed to calculate this dimension and the related quantities ([L1,2], [LW1], [LN2], [JRS], [STZ]).

The second recent development concerns the relationship between the local dimension and the L^q-dimension of a measure, which has been motivated by some physical models. Let μ be a bounded regular Borel measure on $\mathbb{R}^d$ that has compact support. For each $x \in \mathbb{R}^d$, let $B_h(x)$ denote the closed ball of radius h centered at x, and let

$$f(\alpha) = \dim_{\mathcal{H}}\{x : \lim_{h\to 0+} \frac{\ln \mu(B_h(x))}{\ln h} = \alpha\}$$

be the Hausdorff dimension of the set of all x with local dimension α. We call $f(\alpha)$ the *local dimension spectrum* (or *singularity spectrum*) of μ and loosely refer to μ as a *multifractal* measure if $f(\alpha) \neq 0$ for a continuum of α. This spectral parameter was first proposed by physicists to study various multifractal models arising from natural phenomena, e.g., turbulence, diffusion-limited aggregation and percolation (Mandelbrot [M1], Frisch and Parisi [FP], Halsey *et al* [HJ]), and they suggested that the corresponding $f(\alpha)$ should be characteristic quantities. In order to determine the function $f(\alpha)$, Hentschel and Procaccia [HP], Halsey *et al* [HJ] and Frisch and Parisi [FP] introduced the following calculation. Let $\tau(q) = \underline{\lim}_{h\to 0^+} \ln \sum_i \mu(Q_i)^q / \ln h,\ q \in \mathbb{R}$ (this is called the *L^q-spectrum* or *moment scaling exponent* of μ), and let $\tau^*(\alpha) = \inf\{q\alpha - \tau(q) : q \in \mathbb{R}\}$ be the *concave conjugate* of τ (also known as the *Legendre transformation* of τ). They observed the following heuristic relationship between τ and f:

If the measure μ is constructed from the cascade algorithm and if τ and f are smooth and concave, then $\tau^*(\alpha) = f(\alpha)$ (and dually, $f^*(q) = \tau(q)$).

We call such a relationship the *multifractal formalism*; it is also called the *thermodynamic formalism* because of its analogy to the Gibbs state, the pressure and the variational principle in thermodynamics (see Bowen [B], and Bohr and Rand [BR]). The basic mathematical question is to provide appropriate conditions and justifications for the principle, and to reveal the basic structure of the underlying dynamical systems. In a number of cases the principle has been verified rigorously, e.g. the hyperbolic cookie-cutter maps (Rand [R]), the critical maps on the circle with golden rotation number (Collet *et al* [CLP]) and the maximal measures associated with rational maps on the complex plane (Lopes [Lo]). A more substantial advance concerns the multifractal measures defined by the various forms of self-similarity. By now a rather complete theory is known when the iterated function systems consist of similitudes that satisfy certain separation conditions ([AP], [F2], [CM], [EM], [O1,2,3], [Ri1,2]). There are also new developments for non-separated cases [LN1,2] and also for self-similar functions ([J], [DL3]). Despite all these cases, however, the precise range of validity of the multifractal formalism is still not clear.

In this paper we will give an expository survey on self-similar measures centered around the L^q-dimension and the multifractal formalism. The set-up is also tailored to fit into the framework of harmonic analysis . If the IFS satisfies the so-called open set condition of Hutchinson [Hut], then there is an exact formula for the L^q-dimension of the associated measure (Section 3). The situation is more complicated without this condition. A typical case is the infinitely convolved Bernoulli measure (ICBM) which is generated by the IFS: $S_1x = \rho x,\ S_2x = \rho x + 1 - \rho, 1/2 < \rho < 1$ with probability $1/2$ on each map. Unlike the case where $0 < \rho < 1/2$, there are still many unanswered questions for such measures despite the fact that they have been studied since the 30's. In Section 4 we will consider the L^q-dimension of the class of ICBMs that are singular. Our calculations of the L^q-dimension in Section 3 and 4 depend on the L^q-density (see (2.2)). It will be shown that in all cases the densities possess a certain periodic property inherited from (1.1). For the special case $q = 2$, this property has been used to study the Fourier asymptotic average of μ through a Tauberian theorem ([L1], [LW1]). It will be stated in Section 3.

We consider the multifractal formalism in Section 5. After introducing the basic facts about concave and conjugate concave functions, we give some general results for the principle. We also outline the main idea of an elegant theorem of Cawley and Mauldin [CM] on the formalism when the $S_jK, j = 1, \cdots, m$, are disjoint. This condition is stronger than the open set condition. Section 6 aims at weakening that condition. We introduce the *weak separation property* (WSP) on the IFS. The open set condition will imply this property; so do some of the ICBM μ_ρ with $1/2 < \rho < 1$ described above. Under this new assumption the multifractal formalism can be justified. The proof is quite different from the previous case and a sketch of it will also be given.

Another interesting and potentially important aspect of the WSP is attributed to the *scaling functions* defined by the two-scale dilation equations. Such functions are used to generate wavelets and fractal surfaces, and are of great importance in wavelet theory [D] and constructive approximation theory [MP]. The IFS for the scaling function satisfies the WSP. Daubechies and Lagarias [DL3] calculated the L^q-spectrum $\tau(q)$ and the singularity spectrum $f(\alpha)$ for some special scaling functions and showed that the multifractal formalism holds in a certain sense even though the coefficients (corresponding to the probability weights) need not be positive; but no general theory is available yet. This interesting development and the calculation of the L^q-spectrum for solutions to the dilation equation [LM] will be discussed in Section 7.

§2. The L^q-dimension.

Throughout the paper we assume that μ is a positive bounded regular Borel measure on $\mathbb{R}^d$ with bounded support. We will first introduce some definitions of dimension of a measure so as to capture the idea that $\mu(B_h(x))$ behaves like h^α as $h \to 0^+$. The most basic one is the *local dimension* of μ at x defined as

$$\lim_{h\to 0+} \frac{\ln \mu(B_h(x))}{\ln h}, \qquad x \in \operatorname{supp}(\mu).$$

In general we would like to have further global information of the measure, and this can be obtained by L^q-averages and related concepts. For $h > 0$ and $q \in \mathbb{R}$, let $S_h(q) = \sup \sum_i \mu(B_h(x_i))^q$ be the *q-variation* of μ, where the supremum is taken over all disjoint family of closed h-balls $\{B_h(x_i)\}_i$ with $x_i \in \operatorname{supp}(\mu)$. Note that $0 < S_h(q) < \infty$ holds for all $q \in \mathbb{R}$ and for $q = 1$, $C_1 \leq S_h(q) \leq C_2$ for some C_1, C_2 independent of h. We define the *L^q-spectrum* of μ by

$$\tau(q) = \varliminf_{h\to 0+} \frac{\ln S_h(q)}{\ln h}, \qquad q \in \mathbb{R}. \tag{2.1}$$

Let $\underline{\dim}_B(\mathrm{E})$ denote the lower box dimension of a set E. It is easy to show (see for example [LN1]) that

Proposition 2.1. *$\tau : \mathbb{R} \to [-\infty, \infty)$ is an increasing concave function with $\tau(1) = 0$ and $\tau(0) = -\underline{\dim}_B(\operatorname{supp}\mu)$.*

Let $\{Q_h(x_i)\}_i$ be the h-mesh cubes centered at x_i and intersecting supp μ. For $q \geq 0$, $\tau(q)$ has an equivalent and more familiar expression obtained by taking cubes from the h-meshes instead of packing with the disjoint h-balls:

$$\tau(q) = \varliminf_{h\to 0+} \frac{\ln \sum_i \mu(Q_h(x_i))^q}{\ln h}.$$

The equality is false for $q < 0$. However Riedi [Ri1] showed that if the $Q_h(x_i)$'s are replaced by $\widetilde{Q}_h(x_i)$'s which are three times larger and have the same centers at x_i, then the above equality holds for all $q \in \mathbb{R}$. There are two other expressions for $\tau(q)$ which have also been used frequently: let

$$I_h(q) = \int_{\mathbb{R}^d} \mu(B_h(x))^q dx \qquad \text{and} \qquad \tilde{I}_h(q) = \int_{\mathbb{R}^d} \mu(B_h(x))^{q-1} d\mu(x).$$

For $q > 0$, a simple argument shows that $h^{-d} I_h(q)$ and $h^{-d} \tilde{I}_h(q)$ are equivalent to $S_h(q)$ in the sense that their quotients are bounded by constants. It follows that

$$\tau(q) = \varliminf_{h \to 0+} \frac{\ln \int \mu(B_h(x))^q dx}{\ln h} - d = \varliminf_{h \to 0+} \frac{\ln \int \mu(B_h(x))^{q-1} d\mu(x)}{\ln h} - d.$$

For $q > 1$ we define the lower L^q-dimension of μ by

$$\underline{\dim}_q(\mu) = \tau(q)/(q-1),$$

and for $q = 1$

$$\underline{\dim}_1(\mu) = \varliminf_{h \to 0+} \frac{\inf \sum_i \mu(B_h(x_i)) \ln \mu(B_h(x_i))}{\ln h},$$

where $\{B_h(x_i)\}_i$ is a disjoint family of balls as before and the infimum is taken over all such families. For $q = \infty$ and $-\infty$, we let

$$\underline{\dim}_\infty(\mu) = \varliminf_{h \to 0+} \frac{\ln(\sup_x \mu(B_h(x)))}{\ln h}, \quad \underline{\dim}_{-\infty}(\mu) = \varliminf_{h \to 0+} \frac{\ln(\inf_x \mu(B_h(x)))}{\ln h},$$

where the supremum and infimum are taken over all $x \in \operatorname{supp}(\mu)$. By an obvious modification we can define the corresponding upper dimensions, and the dimensions if the limits exist. Note that $\dim_1(\mu)$ is also known as the *entropy* dimension of μ (Rényi [Re1]), $\dim_2(\mu)$ the *correlation* dimension, and L^q-dimension, $q > 1$, the *generalized Rényi dimension* [HP] (in fact all these dimensions have been discussed by Renyi [Re2], Chapter 9 on information theory). Heuristically, $\dim_1(\mu)$ is the limit of $\dim_q(\mu)$ as $q \to 1$ in accordance with l'Hospital's rule (since $\tau(1) = 0$). The following simple proposition is proved in [LN1].

Proposition 2.2. *Let* Dom $\tau = \{q : \tau(q) < \infty\}$. *Then*

(i) Dom $\tau = \mathbb{R}$ *if and only if* $\overline{\dim}_{-\infty}(\mu) < \infty$; Dom $\tau = [0, \infty)$ *if and only if* $\overline{\dim}_{-\infty}(\mu) = \infty$.

(ii) $\lim_{q\to\infty} \underline{\dim}_q(\mu) = \underline{\dim}_\infty(\mu) \le \overline{\dim}_\infty(\mu) \le d.$

Another very useful notion is the (q,α)-*upper density* of μ defined by:

$$\overline{D}^q_\alpha(\mu) = \overline{\lim_{h\to 0+}} \Big(\frac{1}{h^{(d+\alpha(q-1))}} \int \mu(B_h(x))^q dx\Big)^{1/q}. \tag{2.2}$$

It is clear that if $0 < \overline{D}^q_\alpha(\mu) < \infty$, then $\underline{\dim}_q(\mu) = \alpha$ (note that there is a switch of the lower and upper signs in the two expressions), and if $0 < \underline{D}^q_\alpha(\mu) \le \overline{D}^q_\alpha(\mu) < \infty$, then $\dim_q(\mu) = \alpha$. In the next two sections we will use this density parameter extensively to calculate the L^q-dimension for the self-similar measure.

To conclude this section we recall that the Hausdorff dimension of μ at x is defined by

$$\dim_{\mathcal{H}}(\mu) = \inf\{\dim_{\mathcal{H}} E : \mu(\mathbb{R}^d \setminus E) = 0\}.$$

It is known that (Frostman's Lemma) if the local dimension of μ at x equals α for μ almost all x, then $\dim_{\mathcal{H}}(\mu) = \alpha$; furthermore Young [Y, Theorem 4.4] proved that this α also equals the entropy dimension of μ .

§3. Self-similar measures and the open set condition.

Unless otherwise stated we assume that $\{S_j\}_{j=1}^m$ is an IFS of contractive similitudes. For any fixed $m \in \mathbb{N}$, we use $J = (j_1, \cdots, j_k)$, where $j_i \in \{1, \cdots, m\}, i = 1, \cdots k$, $k \in \mathbb{N}$, to denote the multi-index and $|J| = k$ its length. Also we let

$$S_J = S_{j_1} \circ \cdots \circ S_{j_k} \quad \text{and} \quad c_J = c_{j_1} \cdots c_{j_k}$$

for any $\{c_j\}_{j=1}^m$ in $\mathbb{R}$. By using the contraction principle it is easy to show that there exists a compact subset K in $\mathbb{R}^d$ invariant under S_j, i.e. $K = \bigcup_j S_j(K)$. We say that $\{S_j\}_{j=1}^m$ satisfies the *open set condition* [Hut] if there exists a bounded nonempty open set U(called the *basic open set*) such that

$$S_i(U) \subseteq U \quad \text{and} \quad S_i(U) \cap S_j(U) = \emptyset \quad \forall\, i \ne j. \tag{3.1}$$

Under this condition, the invariant set K is contained in $\overline{U}$ and $\bigcup_{|J|=k} S_J(\overline{U})$ for each k. Moreover each component $K \cap \overline{U}$ of K is an identical copy (up to a scaling multiple) of the set K. Let $\mathcal{H}_\alpha$ denote the α-Hausdorff measure. Suppose s satisfies $\sum_{i=1}^m \rho_i^s = 1$, then $0 < \mathcal{H}_s(K) < \infty$. Such s is called the *similarity dimension* of K, it is elementary to show that it equals the Hausdorff dimension of K.

The open set condition plays a crucial role in the self-similar structure of K. Once a basic open set U is given we can construct trivially another basic open set V such that $V \cap K = \emptyset$ by taking $V = U \setminus K$. An important question is whether there exists a basic open set V such that $V \cap K \ne \emptyset$. This is answered by the following theorem of Schief [Sc] (see also [BG]).

Theorem 3.1. *The following statements are equivalent:*

(i) $\{S_j\}_{j=1}^m$ *satisfies the open set condition;*

(ii) There exists a basic open set U *such that* $U \cap K \neq \phi$*;*

(iii) $\mathcal{H}_s(K) > 0$ *where* s *satisfies* $\sum_{j=1}^m \rho_j^s = 1$.

Let $\mu = \sum_{j=1}^m a_j \mu \circ S_j^{-1}$ be the self-similar measure defined by (1.1). We will first consider the support of μ with respect to the basic open set U under the open set condition. In [LW1] the following dichotomy result was proved: either $\mu(U) = 1$ or $\mu(U) = 0$. Theorem 3.1 yields

Corollary 3.2. *Suppose* $\{S_j\}_{j=1}^m$ *satisfies the open set condition, then there exists a basic open set* U *such that* $\mu(U) = 1$ *for any self-similar measure* μ *defined by (1.1). Equivalently* $\mu(\partial U) = 0$ *where* ∂U *denotes the boundary of* U.

Proof. Let U be chosen as in Theorem 3.1 (ii). Let $x \in U \bigcap K$, then there exists a sequence of indices $\{j_n\}$ such that $\{x\} = \bigcap_n S_{J_n}(\overline{U})$, where $J_n = (j_1, \cdots, j_n)$. It follows that $S_{J_n}(\overline{U}) \subseteq U$ for some n. Note that

$$\mu(S_{J_n}(\overline{U})) = \sum_{|J|=n} a_J S_J^{-1}(S_{J_n}(\overline{U})) \geq a_{J_n}\mu(\overline{U}) > 0.$$

This implies that $\mu(U) > 0$ and by the dichotomy criterion, $\mu(U) = 1$.

The corollary was also proved by Graf [Gr]. We remark that $\mu(\partial U) = 0$ is a technically important condition and has been used in several proofs (Theorem 3.3-6 below) in previous papers where it was assumed in addition to the open set condition. Corollary 3.2 implies that such an assumption is redundant.

For a given set of probability weights $\{a_i\}_{i=1}^m$, we call

$$\alpha = \sum_{j=1}^m a_j \log a_j \Big/ \sum_{j=1}^m \log \rho_j$$

the *similarity dimension* of μ. If s satisfies $\sum \rho_j^s = 1$, then $a_j = \rho_j^s$, $j = 1, \cdots, m$ maximizes the above α, with maximum value s. In this case μ equals a constant multiple of $\mathcal{H}_s$ restricted to K and $\{\rho_j^s\}$ is called the *natural weights* for the family $\{S_j\}_{j=1}^m$. For general weights, we have the following theorem.

Theorem 3.3. *Let* μ *be a self-similar measure defined by* $\{S_j\}_{j=1}^m$ *and suppose* $\{S_j\}_{j=1}^m$ *satisfies the open set condition. Let* α *be the similarity dimension and*

$$G = \{x : \lim_{h\to 0^+} \frac{\ln \mu(B_h(x))}{\ln h} = \alpha\}. \tag{3.2}$$

Then μ *is concentrated in* G *and* $\dim_{\mathcal{H}} G = \dim_{\mathcal{H}}(\mu) = \dim_1(\mu) = \alpha(\leq s)$.

The theorem was first proved by Geromino and Hardin [GH] (and implicitly by Cawley and Mauldin [CM]) using the ergodic theorem, Frostman's lemma and a theorem of Young, and assuming a separation condition stronger than the open set condition. It was also proved by Strichartz [Str2] using Corollary 3.2 and the law of iterated logarithm to construct the G.

We next consider the quotient $h^{-(d+\alpha(q-1))}\int \mu(B_h(x))^q dx$ in the definition of (q,α)-density (2.2). This quotient inherits the self-similar property from (1.1) and exhibits periodic behavior as $h \to 0^+$ when α is appropriately chosen. Specifically we have:

Theorem 3.4. *Let μ be a self-similar measure defined by $\{S_j\}_{j=1}^m$ and suppose $\{S_j\}_{j=1}^m$ satisfies the open set condition. For $q>0$, the L^q-spectrum $\tau(q)$ satisfies*

$$\sum_{j=1}^m a_j^q \rho_j^{-\tau(q)} = 1. \tag{3.3}$$

Furthermore, if we let $\varphi(h) = \frac{1}{h^{d+\tau(q)}}\int_{\mathbb{R}^d} \mu(B_h(x))^q dx$, then

$$\varphi(h) = p(h) + o(h) \quad \text{as } h \to 0^+ \tag{3.4}$$

where $p>0$ is a non-zero constant if $\{-\ln \rho_j\}_{j=1}^m$ is non-arithmetic; $p(\lambda t) = p(t)$ if $\{-\ln \rho_j\}_{j=1}^m$ is arithmetic and λ is the least common divisor.

Theorem 3.4 was proved in [LW1] (for the case $q=2$). In the following we outline the proof for the case $q>0$. This will serve as an illustration of how the well known renewal equation in (3.6) [Fe] can be applied to calculate the dimension. In [La] Lalley also used the renewal equation to calculate the packing dimension in another application.

First let us assume the stronger separation condition that the basic open set U satisfies

$$S_i(\overline{U}) \cap S_j(\overline{U}) = \emptyset, \quad i \neq j, \tag{3.5}$$

so that for h sufficiently small, $\{S_j^{-1}(B_h(x))\}_{j=1}^m$ are mutually disjoint. For brevity we write $\mu \circ S_j^{-1} = \mu_j$ and $\tau(q) = \alpha$. By (1.1),

$$\begin{aligned}
\varphi(h) &= \frac{1}{h^{d+\alpha}} \int \Big(\sum_j a_j \mu_j(B_h(x))\Big)^q dx \\
&= \frac{1}{h^{d+\alpha}} \sum_j a_j^q \int \mu_j(B_h(x))^q dx \qquad \text{(by disjointness)} \\
&= \frac{1}{h^{d+\alpha}} \sum_j a_j^q \rho_j^d \int \mu(B_{\frac{h}{\rho_j}}(x))^q dx \qquad \text{(change of variable)} \\
&= \sum_j a_j^q \rho_j^{-\alpha} \varphi\Big(\frac{h}{\rho^j}\Big).
\end{aligned}$$

for sufficiently small h. It follows that for $h > 0$, $\varphi(h) = \sum_j a_j^q \rho_j^{-\alpha} \varphi(\frac{h}{\rho^j}) + o(h)$. By a change of variables the functional equation can be adapted to the following convolution equation (*renewal equation*)

$$f(x) = \int_0^x f(x-y)d\sigma(y) + S(x), \quad x \geq 0 \tag{3.6}$$

where $S(x)$ is a continuous integrable function with $\lim_{x\to\infty} S(x) = 0$. In order to have a nontrivial bounded solution, σ must be a probability measure. This implies that $\sum a_j^q \rho_j^{-\alpha} = 1$ and the expression of φ in Theorem 3.4 follows from the known solution of the renewal equation [Fe].

Without the stronger condition in (3.5) we have to modify the second identity by

$$\varphi(h) = \frac{1}{h^{d+\alpha}} \sum_j a_j^q \int \mu_j(B_h(x))^q dx + e(h)$$

where

$$e(h) \leq \frac{2}{h^{d+\alpha}} \int_{\bigcup_j \{x:\ \mathrm{dist}(x,\partial U_j)<h\}} (\sum_j a_j \mu_j (B_h(x))^q dx.$$

The estimation of the error term requires more work, it is the place we need to choose the basic open set U such that $\mu(\partial U) = 0$ as remarked at the end of Corollary 3.2. The technicalities of the estimation can be found in [LW1].

There are various applications of the asymptotic formula for $\varphi(h)$ [Str6]. The most interesting case is for $q = 2$ and Theorem 3.4 yields an asymptotic average behavior of the Fourier transformation of the measure. This is a consequence of an extension of Wiener's Tauberian Theorem proved in [LW1]:

Theorem 3.5. *Let μ be any bounded regular Borel measure in $\mathbb{R}^d$. Then the following two expressions are equivalent:*

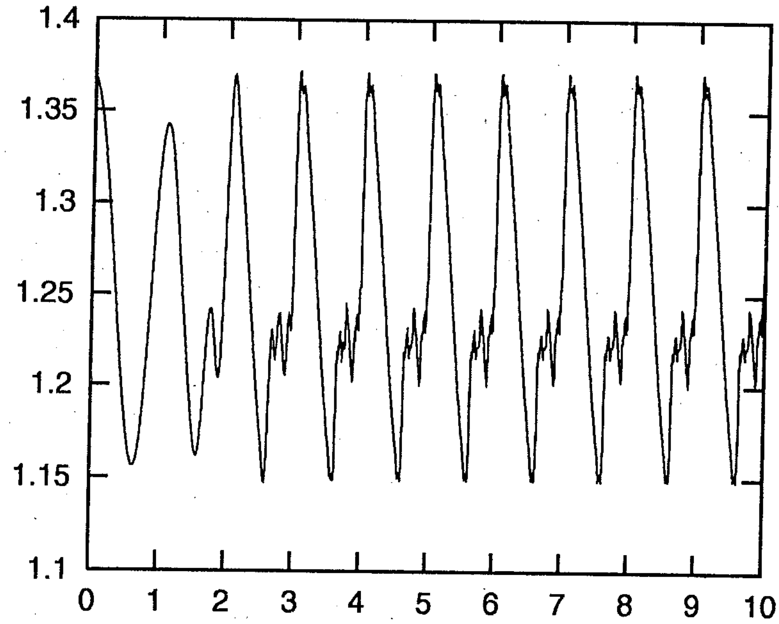
$$\lim_{h\to 0^+} \left[\frac{1}{h^{d+\alpha}} \int \mu(B_h(x))^2 dx - p(h) \right] = 0,$$

$$\lim_{T\to\infty} \left[\frac{1}{T^{d-\alpha}} \int_{|\xi|\leq T} |\hat{\mu}(\xi)|^2 d\xi - q(T) \right] = 0,$$

where p, q are multiplicative periodic functions with the same period.

For more information on this the reader can refer to [JRS], [Str1-6]. As a simple example, consider the standard Cantor measure μ. Its Fourier transformation is $\hat{\mu}(\xi) = e^{i\xi/2} \prod_{k=1}^{\infty} \cos(\xi/3^k)$. It was observed by Wiener and Wintner [WW] that $\hat{\mu}(\xi)$ is uniformly bounded by $\log 2/\log 3$, and there is no limit as $|\xi| \to \infty$. The following picture is the the graph of $\psi(T) =$

$T^{-(1-\alpha)}\int_{|\xi|\le T}|\hat{\mu}(\xi)|^2 d\xi$ where $\alpha = \log 2/\log 3$, and the horizontal coordinate is plotted on the $\log_3 T$ scale instead of T.



There is an important vector-valued extension of self-similar measures. In (1.1) the measure μ is a linear combination of pieces that are similar to μ, the vector-valued version says that each measure in the family (i.e. each coordinate measure) is a linear combination of pieces similar to the other measures in the family. This can be formulated in terms of a very general directed graph device introduced by Mauldin and Williams [MW] and many others (e.g., Bandt, Barnsley, Berger, Bedford, Culik and Dube, Dekking). Let (V, E) denote a directed graph where V is a set of vertices and E is the set of edges such that for each $u \in V$, there are some edges $e \in E$ going out from u. Let E_u denote the set of edges from u and let E_{uv} be the set of edges from u to v. A *path* is a sequence of edges $\gamma = (e_1, \cdots e_n)$ where $e_j \in E_{u_j u_{j+1}}$ for some sequence $\{u_1, \cdots, u_{n+1}\} \subseteq V$ and a *cycle* if $u_1 = u_{n+1}$.

Suppose $\{S_e\}_{e\in E}$ is a family of similitudes on $\mathbb{R}^d$ such that the similarity ratios $\{r(e)\}_{e\in E}$ satisfy $r(\gamma) = r(e_1)\cdots r(e_n) < 1$ for each cycle $\gamma = (e_1, \cdots, e_n)$, then there exists a unique family of compact sets $\{K_v\}_{v\in V}$ satisfying

$$K_v = \bigcup_{u\in V}\bigcup_{e\in E_{uv}} S_e(K_u) \qquad \text{for all} \quad v \in V.$$

This was called a *graph-directed construction* by Mauldin and Williams [MW], [EM] (some other names are mixed self-similar set or recurrent IFS). The open set condition of $\{S_e\}_{e\in E}$ can be defined as follows: there exists a family of bounded nonempty open sets $\{U_v\}_{v\in V}$ such that

$$S_e(U_u) \subseteq U_v \qquad \text{for all} \quad e \in E_{uv}$$

and $\{S_e(U_u)\}$ are disjoint for u running through V and $e \in E_{uv}$. In this case the measure μ_u is supported by $\overline{U}_u$. In [W] Wang extended Theorem 3.1 that such $\{U_v\}_{v\in V}$ can be chosen so that $U_v \cap K_v \neq \emptyset$, provided that the graph (V, E) is strongly connected (i.e. each $u, v \in V$ can be joined by a path from u to v). (The statement is false without the strong connectedness.)

We can also define a self-similar family of positive measures $\{\mu_v\}_{v\in V}$ as

$$\mu_v = \sum_{u\in V} \sum_{e\in E_{uv}} p(e)\mu_u \circ S_e^{-1} \quad \text{for all} \quad v \in V \tag{3.7}$$

where the weights $p(e)$ are positive and satisfy

$$\sum_{u\in V} \sum_{e\in E_{uv}} p(e) = 1 \quad \text{for all} \quad v \in V.$$

Assuming the open set condition and using a similar argument in the proof of Theorem 3.4, it was proved in [Str4] that if (V, E) is strongly connected and $\{S_e\}_{e\in E}$ satisfies the open set condition, then the L^q-spectrum $\tau(q), q > 0$ of the family $\{\mu_v\}_{v\in V}$ is the positive number such that the matrix

$$\Big[\sum_{e\in E_{uv}} p(e)^q r(e)^{-\tau(q)} \Big]_{u,v\in V} \tag{3.8}$$

has maximal eigenvalue 1. ((3.8) was first used by Edgar and Mauldin [EM] in their set up of multifractal formalism.) Moreover the density quotient $\varphi_v(h) = h^{-(d+\tau(q))} \int \mu_v(B_h(x))^q dx$ has an asymptotic multiplicative periodic property as in Theorem 3.4. The exact expression can be found in [Str4] and [LWC].

The notion of a self-similar family of measures can be applied to consider some nonlinear IFS. A *conformal map* is a C^1 map whose differential at each point is a similitude. By replacing the similitudes with the conformal maps, it is easy to extend the definition of self-similar measures to *self-conformal* measures. This includes the important example of measures on Julia sets where the mappings S_j are the local inverses of a rational function. To find the L^p-dimension of a self-conformal measure, one can cut the support of μ into small pieces to form a family of self-conformal measures, which is approximately self-similar. The dimension of the original μ can then be approximated by using (3.7) on the new family [Str4]. Similar method has also been dicussed in [F1, Section 9.3] for calculating the box dimension and Hausdorff dimension of the corresponding self-conformal set.

Recently Mauldin and Urbanski [MU] have extended the self-conformal measures to allow the IFS to have infinitely many conformal maps.

Our construction of self-similar measures (1.1) depends on the IFS, i.e., it is a map-specifying construction. There is a simple non-map-specifying construction called the Moran construction [CM]: Let E be a nonempty subset such that $(E^\circ)^- = E$ (E°, E^- means the interior and the closure of E respectively). Let $E_0 = E$ and suppose we have constructed E_J with $J = (j_1, \cdots, j_n)$, $j_i \in \{1, \cdots, m\}$, let $E_{(J,j)} \subseteq E_J$ such that E and $E_{(J,j)}$ has similarity ratio $0 < \rho_j < 1$. The invariant set $K = \bigcap_n \bigcup_{|J|=n} E_J$ is called the *Moran fractal.* If we associate probability weights a_j with each j, we will have an invariant probability measure μ. Theorem 3.3 will hold under the condition that at each level n, the family $\{E_J : |J| = n\}$ is disjoint. The open set condition can also be adjusted by assuming that for each n, $\{E_J^\circ : |J| = n\}$ is disjoint. However in this case we do not know whether the other theorems in this section can be extended.

§4. Bernoulli convolutions.

For $0 < \rho < 1$, let $S_1(x) = \rho\, x$, $\quad S_2(x) = \rho\, x + (1 - \rho)$, $\quad x \in \mathbb{R}$ and let $\mu (= \mu_\rho)$ be the self-similar measure defined by

$$\mu = \frac{1}{2}\mu \circ S_1^{-1} + \frac{1}{2}\mu \circ S_2^{-1}. \tag{4.1}$$

For $0 < \rho < \frac{1}{2}$, μ is the Cantor type measure which is the most fundamental self-similar measure. For $\frac{1}{2} < \rho < 1$ the measure is more mysterious; the basic difference is that it does not satisfy the open set condition any more and there is overlapping on the attractor $[0, 1]$ under the two similitudes. This creates a lot of complication; it is not known when the measure will be absolutely continuous or singular, or how to calculate the dimensions when it is singular. In this section we will discuss some recent work in this direction. The techniques may be useful as canonical approaches to handle IFS that do not satisfy the open set condition.

First we see that the measure can be identified with the infinite Bernoulli convolution as follows: Let $\{X_n\}_{n=1}^\infty$ be a sequence of i.i.d. Bernoulli random variables (i.e. X_n takes values $\{0, 1\}$ with probability $\frac{1}{2}$ each). For $0 < \rho < 1$ let $X = (1-\rho)\sum_{n=0}^\infty \rho^n X_n$ and let μ be the corresponding distribution, then μ is the infinite convolution of the sequence $\{\mu_n\}_{n=1}^\infty$ where μ_n is the point mass measure concentrated at 0 and $(1 - \rho)\rho^n$ with weights $\frac{1}{2}$ each. Following the notation of Alexander and Yorke [AY] we call such μ an *infinitely convolved Bernoulli measure* (ICBM). Note that the Fourier transformation of μ defined here and in (4.1) are both equal to $e^{i(1-\rho)\xi/2} \prod_{n=1}^\infty \cos(\rho^n(1 - \rho)\xi)$; and hence they define the same measure.

It is easy to show that if $\rho = 2^{-1/k}, k = 1, 2, \cdots$, then μ is absolutely continuous. More fascinating results are known for some classes of algebraic integers. Let $\beta = \rho^{-1}$ and let $\beta_1, \cdots, \beta_m$ denote the algebraic conjugate of β. Erdös and Salem [S] proved that β is a P.V. number (i.e. $|\beta| > 1$ and $|\beta_i| < 1, i = 1, \cdots, m$) if and only if $\hat{\mu}(\xi) \not\to 0$ as $|\xi| \to \infty$, in this case μ is singular. On the other hand Garsia [G] showed that if $\beta \prod_{|\beta_i|>1} \beta_i = 2$, then μ is absolutely continuous. Note that in this case it is necessary that $|\beta_i| > 1$ for all $i = 1, \cdots, m$.

A classification of ρ for μ_ρ to be absolutely continuous or singular remains widely open, in particular it is not known even for the case when ρ is a rational number. In another direction Erdös proved that for almost all ρ sufficiently close to 1, then ρ is absolutely continuous [E]. He conjectured that the result should also be true for almost all $1/2 < \rho < 1$. This has also been recently solved positively by Solomyak [So].

In the following we discuss the L^q-dimension of the ICBM $\mu(= \mu_\rho)$ with ρ^{-1} a P.V. number. We will apply the self-similar identity (4.1) to the quotient $\varphi(h) = h^{-(1+\alpha)} \int \mu(B_h(x))^q dx$ as in Theorem 3.4. However the technique is quite different now because the open set condition is not satisfied here. We are able to calculate the L^q-dimension of such μ when $q > 1$ is an integer. For the special case $\rho = (\sqrt{5} - 1)/2$, we have an exact formula that works for all $q \geq 0$.

For convenience we consider the case $q = 2$ first. We let

$$\Phi_\gamma^{(\alpha)}(h) = \frac{1}{h^{1+\alpha}} \int_\gamma \mu(B_h(x))\mu(B_h(y))$$

where

$$\gamma : \begin{cases} x = t + a, \\ y = t, \end{cases} \qquad -\infty < t < \infty$$

denotes the line parallel to the diagonal with x-intercept at a, and $\int_\gamma$ denotes the line integral on γ. Since μ is concentrated on a dense subset of [0,1] we see that the "effective domain" of the above integral is on $\gamma \cap ([0,1] \times [0,1])$ (see the following figure).

Lemma 4.1. *Suppose γ has an $x-$intercept at a.*

(i) If γ' has $x-$intercept at $-a$, then $\Phi_\gamma^{(\alpha)}(h) = \Phi_{\gamma'}^{(\alpha)}(h)$.

(ii) $a \notin [-1, 1]$ if and only if there exists $h_0 > 0$ such that $\Phi_\gamma^{(\alpha)}(h) = 0$ for all $0 < h < h_0$.

(iii) For $a = 1$ or -1, there exists $\delta > 0$ such that for $0 < \alpha \leq 1$, $\Phi_\gamma^{(\alpha)}(h) = o(h^\delta)$ as $h \to 0^+$.

Let $\triangle$ be the set of "basis elements" that spans a real vector space $\langle \triangle \rangle$. It follows from a simple line integral argument that

$$\Phi_{c_1\gamma_1+\cdots+c_n\gamma_n}^{(\alpha)}(h) = c_1\Phi_{\gamma_1}^{(\alpha)}(h) + \cdots + c_n\Phi_{\gamma_n}^{(\alpha)}(h).$$

By substituting the expression for μ in (4.1) into $\Phi_\gamma^{(\alpha)}(h)$ (as in the proof of Theorem 3.4), we have

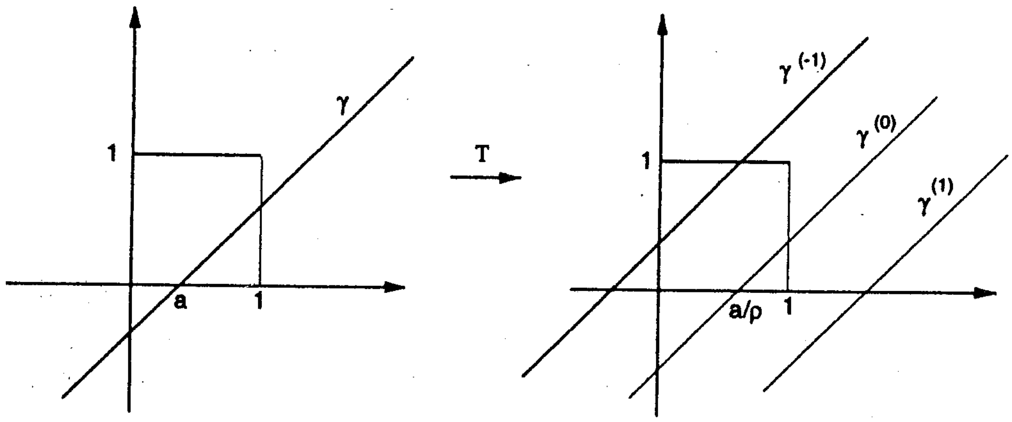
$$\begin{aligned}\Phi_\gamma^{(\alpha)}(h) &= \frac{1}{4\rho^\alpha}\left[\Phi_{\gamma^{(-1)}}^{(\alpha)}\left(\frac{h}{\rho}\right) + 2\Phi_{\gamma^{(0)}}^{(\alpha)}\left(\frac{h}{\rho}\right) + \Phi_{\gamma^{(1)}}^{(\alpha)}\left(\frac{h}{\rho}\right)\right] \\ &= \frac{1}{4\rho^\alpha}\Phi_{\gamma^{(-1)}+2\gamma^{(0)}+\gamma^{(1)}}^{(\alpha)}\left(\frac{h}{\rho}\right) \end{aligned} \tag{4.2}$$

where

$$\gamma^{(\epsilon)}: \begin{cases} x = t + \left(\dfrac{a}{\rho} + \epsilon\dfrac{1-\rho}{\rho}\right), \\ y = t, \end{cases}$$

for $\epsilon = -1, 0$ and 1. Let $T : \langle\triangle\rangle \to \langle\triangle\rangle$ be defined by $T(\gamma) = \gamma^{(-1)} + 2\gamma^{(0)} + \gamma^{(1)}$(see the figure), then we can rewrite (4.2) into the following basic relationship

Proposition 4.2. *For $0 \le \alpha \le 1$, $h > 0$, we have $\Phi_\gamma^{(\alpha)}(h) = \frac{1}{4\rho^\alpha}\Phi_{T(\gamma)}^{(\alpha)}(\frac{h}{\rho})$.*



Let $\epsilon_0 = 0$, $\gamma^0 = \gamma^{(\epsilon_0)} \in \triangle$ be the line with $x-$intercept at 0, and let $\gamma^{\epsilon_0\cdots\epsilon_n} = \left(\gamma^{\epsilon_0\cdots\epsilon_{n-1}}\right)^{(\epsilon_n)}$ where $\epsilon_n = -1, 0$ or 1. In this case the $x-$intercept of $\gamma^{\epsilon_0\cdots\epsilon_n}$ is $\frac{1-\rho}{\rho}\sum_{i=0}^{n-1}\epsilon_{n-j}\rho^{-j}$. Let

$$\Gamma = \left\{\gamma^{\epsilon_0\cdots\epsilon_n} : n \in \mathbb{N}\right\}, \quad \Gamma_0 = \left\{\gamma \in \Gamma : \gamma \text{ has } x\text{−intercept at } (-1,1)\right\}.$$

Let $\beta = \rho^{-1}$, then Γ and Γ_0 can be identified with the following sets in $\mathbb{R}$:

$$W = \left\{s : s = \sum_{j=0}^{n-1}\epsilon_{n-j}\beta^j,\ n \in \mathbb{N}\right\}, \quad W_0 = \left\{s \in W : |s| < \frac{1}{\beta - 1}\right\}.$$

We can regard W as the states of the family of random paths such where the n-th state is

$$s^{(n)} = \beta s^{(n-1)} + \epsilon_n = \sum_{j=0}^{n-1} \epsilon_{n-j}\beta^j.$$

It is easy to show that once the path steps outside the interval $\left(-\frac{1}{\beta-1}, \frac{1}{\beta-1}\right)$, then it will never return to the interval. Inside the interval there may be finitely or infinitely many states. For our case we have

Proposition 4.3. *If $\rho^{-1} = \beta$ is a P.V. number, then W_0 is a finite set.*

An elementary proof of this is given in [L2, Theorem 2.5]. It also follows from a lemma of Garsia [G, Lemma 1.51] that there exists $c > 0$ such that the distance between two members in W is at least c. The converse of the proposition is not known and it seems to be a very interesting question.

Under the P.V. number assumption, $\langle\Gamma_0\rangle$ is a finite dimensional linear subspace, and T is invariant on $\langle\Gamma_0\rangle$, by the remark before Proposition 4.3. Together with Proposition 4.2 and Lemma 4.1 (iii), the map $T : \langle\Gamma_0\rangle \to \langle\Gamma_0\rangle$ satisfies

$$\Phi_\gamma^{(\alpha)}(h) = \frac{1}{4\rho^\alpha}\Phi_{T(\gamma)}^{(\alpha)}\left(\frac{h}{\rho}\right) + o(h^\delta).$$

Let λ be the maximal eigenvalue of T with eigenvector $\overline{\gamma}$. Furthermore, let $\alpha = \ln(\lambda/4)/\ln\rho$. Then

$$\Phi_{\overline{\gamma}}^{(\alpha)}(h) = \Phi_{\overline{\gamma}}^{(\alpha)}\left(\frac{h}{\rho}\right) + o(h^\delta). \tag{4.3}$$

It follows that $\Phi_{\overline{\gamma}}^{(\alpha)}(h)$ is continuous and asymptotically multiplicatively periodic with period ρ. From this we have [L2]

Theorem 4.4. *Suppose $1/2 < \rho < 1$ and $\rho^{-1} = \beta$ is a P.V. number. Let λ be the maximal eigenvalue of $T : \langle\Gamma_0\rangle \to \langle\Gamma_0\rangle$ and let $\alpha = \ln(\lambda/4)/\ln\rho$. Then $0 < \alpha < 1$ and $\dim_2(\mu) = \alpha$.*

As a simple corollary we see that μ is singular. This offers a different proof from that of Erdös and Salem through the Fourier transformation. Observe that in the above, $\overline{\gamma} = \sum_{\gamma_i \in \Gamma_0} c_i\gamma_i$ for some $c_i \geq 0$, and that γ^0 is in Γ_0.

If T is irreducible, then $c_i > 0$ for all i and we can use (4.3) to show that $\Phi_{\gamma^0}^{(\alpha)}(h) = h^{-(1+\alpha)}\int \mu(B_h(x))^2 dx$ is multiplicatively periodic with period ρ as $h \to 0^+$; and thus so is the Fourier average $\Psi^{(\alpha)}(r) = r^{-(1-\alpha)}\int_{-r}^{r} \left|\hat{\mu}(\xi)\right|^2 d\xi$ as $r \to \infty$ by Theorem 3.5. In all the examples for P.V. numbers that we have calculated (see below), the operator T is irreducible. We do not know whether this holds in all cases.

We remark that by Lemma 4.1(i) we can identify γ_{-a} with γ_a in the expression of $\Phi_\gamma^{(\alpha)}(h)$. We can hence reduce the size of Γ_0 by considering instead

$$\Gamma_0^+ = \{\gamma : \gamma \text{ has } x\text{-intercept in } [0,1)\},$$

and $T^+ : \langle\Gamma_0^+\rangle \to \langle\Gamma_0^+\rangle$ defined by $T^+ = \pi \circ T$, where π is the natural projection of $\langle\Gamma_0\rangle$ onto $\langle\Gamma_0^+\rangle$. Note that T^+ and T have the same maximal eigenvalue and hence Theorem 4.3 is the same if we replace T by T^+.

We can summarize the above discussion to an algorithm to construct matrices A and A^+ representing T and T^+ respectively:

(I) To find W_0, we start with 0, use induction on $s^{(n)} = \beta s^{(n-1)} + \epsilon_n$, $\epsilon_n = 0, 1$, or -1, and keep those $s^{(n)}$ that are $< \frac{1}{(\beta-1)}$ and distinct from the previously chosen $s^{(1)} \cdots, s^{(n-1)}$.

(II) To construct the matrix A associated with T, we assign to each entry $(t,s) \in W_0 \times W_0$ the number

$$a_{(t,s)} = \begin{cases} 1 & \text{if } t = \beta s + \epsilon,\ \epsilon = \pm 1, \\ 2 & \text{if } t = \beta s + \epsilon,\ \epsilon = 0, \\ 0 & \text{otherwise.} \end{cases}$$

(III) To construct A^+ corresponding to T^+ we truncate the columns that correspond to $s < 0$, and define $a^+_{(t,s)} = a_{(t,s)} + a_{(-t,s)}$ by $(t,s) \in W_0^+ \times W_0^+$.

To illustrate, let $\rho = (\sqrt{5}-1)/2$; so $\beta = (\sqrt{5}+1)/2$. By using $\rho^2+\rho-1=0$, it is easy to show that $W_0 = \{0, 1, \rho, -1, -\rho\}$ and $W_0^+ = \{0, 1, \rho\}$. The matrix representations of T and T^+ are

$$\begin{bmatrix} 2 & 0 & 1 & 0 & 1 \\ 1 & 0 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 2 & 0 & 1 \\ 2 & 0 & 2 \\ 0 & 1 & 0 \end{bmatrix}.$$

By working out the characteristic polynomial of the matrix A^+, we have

Corollary 4.5. *If $\rho = (\sqrt{5}-1)/2$, then $\dim_2(\mu) = \alpha$ where α satisfies*

$$(4\rho^\alpha)^3 - 2(4\rho^\alpha)^2 - 2(4\rho^\alpha) + 2 = 0.$$

The following is the list of P.V. numbers for which $\dim_2(\mu)$ has been calculated, using Theorem 4.3. The minimal polynomial is the defining equation for $\beta = \rho^{-1}$. (The fourth row is the golden number.)

Min. Polynomial	ρ	Size of A^+	λ	$\dim_2(\mu_\rho)$
$x^5-x^4\cdots-1=0$	0.5086604	6	2.0573712	0.9835654
$x^4-x^3\cdots-1=0$	0.5187901	5	2.1118009	0.9733295
$x^3-x^2-x-1=0$	0.5436890	4	2.2226941	0.9642200
$x^2-x-1=0$	0.6180334	3	2.4811943	0.9923994
$x^3-x^2-1=0$	0.6823275	25	2.7302333	0.9991163
$x^4-x^3-1=0$	0.7244919	627	2.8979776	0.9999895
$x^3-x-1=0$	0.7548776	90	3.0195190	0.9999901

For $q > 2$, there are some slight complications with regard to Lemma 4.1(i), (ii), and so we make the following modifications. Let

$$\gamma\ (=\gamma_{\boldsymbol{a}}):\ \begin{cases} x_1 = t + a_1, \\ \vdots \\ x_q = t + a_q, \end{cases} \qquad -\infty < t < \infty$$

be a line in $\mathbb{R}^q$ parallel to the diagonal. Note that $\gamma_{\boldsymbol{a}}$ does not define the line uniquely: $\gamma_{\boldsymbol{a}} = \gamma_{\boldsymbol{b}}$ if and only if $\boldsymbol{a} - \boldsymbol{b} = c(1, \cdots, 1)$ for some $c \in \mathbb{R}$. Let $\boldsymbol{\epsilon} = (\epsilon_1, \cdots, \epsilon_q)$ with $\epsilon_i = 0$ or 1, and let $\gamma^{(\boldsymbol{\epsilon})}$ be the line with $x_i = t + (\frac{a_i}{\rho} + \epsilon_i \frac{1-\rho}{\rho})$, $1 \le i \le q$. Also let $\boldsymbol{\epsilon}^0 = 0$, $\gamma^{\boldsymbol{\epsilon}^0} = \gamma_0$ be the diagonal line, and let $\gamma^{\boldsymbol{\epsilon}^0 \cdots \boldsymbol{\epsilon}^n}$ be defined inductively. On the set

$$\Gamma_0 = \{\gamma = \gamma^{\boldsymbol{\epsilon}^0 \cdots \boldsymbol{\epsilon}^n} \ :\ \gamma \cap (0,1)^q \ne \emptyset,\ n \in \mathbb{N}\},$$

we define $T : \langle \Gamma_0 \rangle \to \langle \Gamma_0 \rangle$ by $T(\gamma) = \sum_{\boldsymbol{\epsilon}} \gamma^{(\boldsymbol{\epsilon})}$. Now let

$$\Phi_\gamma^{(\alpha)}(h) = \frac{1}{h^{1+\alpha}} \int_\gamma \mu\big(B_h(x_1)\big) \cdots \mu\big(B_h(x_q)\big).$$

By substituting the self-similar identity (4.1) of μ and using a change of variables as before, we have the basic identity

$$\Phi_\gamma^{(\alpha)}(h) = \frac{1}{2^q \rho^\alpha} \Phi_{T(\gamma)}^{(\alpha)}\Big(\frac{h}{\rho}\Big) + o(h^\delta).$$

If $\rho^{-1} = \beta$ is a P.V. number, then $\langle \Gamma_0 \rangle$ is a finite set and we can find the maximal eigenvalue λ of T on $\langle \Gamma_0 \rangle$. It follows from the same argument as in Theorem 4.4 that

$$\tau(q) = \frac{\ln(\lambda/2^q)}{\ln \rho}, \quad \dim_q(\mu) = \frac{\tau(q)}{q-1}.$$

For a given $\gamma = \gamma_{\boldsymbol{a}}$, let γ' be obtained by permuting the coordinates of $\boldsymbol{a}$, it is clear that $\Phi_{\gamma}^{(\alpha)}(h) = \Phi_{\gamma'}^{(\alpha)}(h)$. We can use this property to reduce T to T^{σ} defined on

$$\Gamma_0^{\sigma} = \{\gamma \in \Gamma \ : \ \gamma = \gamma_{\boldsymbol{a}},\ a_1 \geq a_2 \cdots \geq a_q = 0\}$$

and preserve the maximal eigenvalue. To construct a matrix A^{σ} to represent T^{σ}, we first identify $\gamma \in \Gamma_0$ with its intercept on the plane $x_q = 0$. The corresponding set is

$$W_0 = \Big\{\boldsymbol{s} = (s_1, \cdots, s_{q-1}, 0) : s_j = \sum_{i=0}^{n-1} \epsilon_j^{(n-i)} \beta^i,\ \epsilon_j^i = -1, 0 \text{ or } 1\Big\} \cap C$$

where $C = \{(s_1, \cdots, s_{q-1}, 0) : |s_i| < \frac{1}{(\beta-1)},\ |s_i - s_j| < \frac{1}{(\beta-1)},\ 0 \leq i, j \leq q-1\}$ is the projection of the open cube $(0, \frac{1}{(\beta-1)})^q$ onto the $x_q = 0$ plane along the diagonal. Let

$$W_0^{\sigma} = \{\boldsymbol{s} \in W_0 : s_1 \geq \cdots \geq s_{q-1} > 0\}.$$

We can use the following algorithm to construct the matrix A^{σ}:

(I) Starting from $\boldsymbol{0}$, suppose we have constructed $\boldsymbol{s} \in W_0^{\sigma}$ in the $(n-1)$th-step. Let $\boldsymbol{t} = \beta \boldsymbol{s} + \boldsymbol{\epsilon}$, $\epsilon_i = 0$ or 1, $1 \leq i \leq q$. Rearrange $\boldsymbol{t}$ to $\boldsymbol{t}_{\sigma}$ so that $t_{\sigma(1)} \geq t_{\sigma(2)} \cdots \geq t_{\sigma(q)}$ and let $\boldsymbol{s}' = \boldsymbol{t}_{\sigma} - t_{\sigma(q)}(1, \cdots, 1)$, then keep the $\boldsymbol{s}'$ that is in W_0^{σ} and is distinct from those previously chosen. (If there is no new number the process is finished.)

(II) For the column of the matrix A^{σ} corresponding to $\boldsymbol{s}$, we assign, for each $\boldsymbol{s}' \in W_0^{\sigma}$, the number of appearances of the $\boldsymbol{t}$ that gives $\boldsymbol{s}' \in W_0^{\sigma}$ in the rearrangement.

As an example we consider $\rho = (\sqrt{5}-1)/2$. By using the above algorithm we found that approximately $\dim_2(\mu) \approx 0.9924$, $\dim_3(\mu) \approx 0.9897$, $\dim_4(\mu) \approx 0.9875$. Also in [Hu], Hu used an algebraic method and showed that $\dim_{\infty}(\mu) = -(1/2) - \log 2/\log \rho \approx 0.9404$. It is seen that for $q \geq 2$ the L^q-dimension lies in a very narrow region close to 1.

Furthermore, for $\rho = (\sqrt{5}-1)/2$, we have an exact formula for the L^q-dimension. It depends on the following reduction of an overlapping case into a nonoverlapping case, due to Strichartz *et al* [STZ]. Let

$$\begin{aligned} T_0 x &= S_1 S_1 x = \rho^2 x, \\ T_1 x &= S_2 S_1 S_1 x = S_1 S_2 S_2 x = \rho^3 x + \rho^2, \\ T_2 x &= S_2 S_2 x = \rho^2 x + (1-\rho^2). \end{aligned}$$

Note that (0, 1) is the disjoint union of $T_i(0,1)$, $i = 1,2,3$ so that the T_i's satisfy the open set condition. On the other hand the self-similar identity (4.1) is reduced to the following "second order" self-similar identities defined by the T_i's: For $A \subseteq [0,1]$,

$$\begin{bmatrix} \mu(T_0T_0A) \\ \mu(T_1T_0A) \\ \mu(T_2T_0A) \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & 0 & 0 \\ \frac{1}{8} & \frac{1}{4} & 0 \\ 0 & \frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} \mu(T_0A) \\ \mu(T_1A) \\ \mu(T_2A) \end{bmatrix}$$

$$\begin{bmatrix} \mu(T_0T_1A) \\ \mu(T_1T_1A) \\ \mu(T_2T_1A) \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{4} & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & \frac{1}{4} & 0 \end{bmatrix} \begin{bmatrix} \mu(T_0A) \\ \mu(T_1A) \\ \mu(T_2A) \end{bmatrix}$$

$$\begin{bmatrix} \mu(T_0T_2A) \\ \mu(T_1T_2A) \\ \mu(T_2T_2A) \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{4} & \frac{1}{8} \\ 0 & 0 & \frac{1}{4} \end{bmatrix} \begin{bmatrix} \mu(T_0A) \\ \mu(T_1A) \\ \mu(T_2A) \end{bmatrix}. \tag{4.4}$$

We denote the three matrices by P_0, P_1, P_2. It follows from (4.4) that for $A \subseteq [0,1]$ and for $J = (j_1, \cdots j_k)$ with $j_i = 0$ or 2,

$$\mu(T_1T_JT_1A) = c_J\mu(T_1A) \quad \text{where} \quad c_J = \frac{1}{4}\,[0,1,0]\; P_J \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}. \tag{4.5}$$

To calculate the L^q-dimension of μ, we will produce a renewal equation for $\int_0^1 \mu(B_h(x))^q dx$ as in the proof of Theorem 3.4.

Observe that

$$\int_0^1 \mu(B_h(x))^q dx = \Big(\int_{T_0[0,1]} + \int_{T_1[0,1]} + \int_{T_2[0,1]}\Big)\mu(B_h(x))^q dx$$

$$=\rho^2 \int_0^1 \mu(B_h(T_0x))^q dx + \rho^3 \int_0^1 \mu(B_h(T_1x))^q dx + \rho^2 \int_0^1 \mu(B_h(T_2x))^q dx. \tag{4.6}$$

Let

$$\Phi_i^{(\alpha)}(h) = \frac{1}{h^{1+\alpha}} \int_0^1 \mu\big(B_h(T_ix)\big)^q dx, \quad i = 0,\ 1,\ 2.$$

Then $\Phi_0^{(\alpha)}(h) = \Phi_2^{(\alpha)}(h)$. By repeating the above argument of splitting the interval [0,1] into three pieces as in (4.6), applying (4.5) and using the change of variables, we have

(i) $\Phi_0^{(\alpha)}(h) = \frac{1}{4\rho^{2\alpha}}\Phi_0^{(\alpha)}(\frac{h}{\rho^2}) + \frac{1}{4\rho^{2\alpha}}\Phi_1^{(\alpha)}(\frac{h}{\rho^2}) + e_0(h)$;

(ii) $\Phi_1^{(\alpha)}(h) = \sum_{k=0}^{\infty}(\sum_{|J|=k} c_J^q)\rho^{-(2k+3)\alpha}\Phi_1^{(\alpha)}(\frac{h}{\rho^{(2k+3)}}) + e(h)$,

where $e_0(h), e(h) = o(h^\delta)$ for some $\delta > 0$. These error terms arise due to the fact that the measure μ on $B_h(T_J(x))$ satisfies different identities on the two sides of the interval when x is a boundary point of $T_J[0,1]$. By using the renewal equation (see Section 3) we conclude that [LN2]:

3. *For $\rho = (\sqrt{5}-1)/2$ and for $0 < q \le a$, then the L^q-spectrum $\tau(q)$ satisfies*

$$\sum_{k=0}^{\infty}\Big(\sum_{|J|=k} c_J^q\Big)\rho^{-(2k+3)\tau(q)} = 1.$$

Moreover $\tau(q)$ is differentiable and the entropy dimension $\dim_1(\mu)$ *is given by*

$$\tau'(1) = \frac{1}{9\ln\rho}\sum_{k=0}^{\infty}\sum_{|J|=k} c_J \ln c_J.$$

Due to some technicalities we are not able to verify the first identity for all $q > 0$ yet. The number 9 in the second expression comes from $\sum_{k=0}^{\infty}(2k+3)\sum_{|J|=k} c_J$. We have checked that for $q = 2$ and 3, the $\tau(q)$'s thus obtained coincide with those from the algorithm in the preceding method. We remark that the entropy dimension of the ICBM for $\rho = (\sqrt{5}-1)/2$ had also been considered in [G], [AY], [AZ], [LP] and the other P.V. numbers in [PU]. The entropy dimension calculated in [AZ] is 0.99692; the calculation from the above formula is close to this number but needs more iterations. Using the above technique to reduce an overlapping case to a nonoverlapping case seems to be quite restrictive. Besides the golden number, another P.V. number has been found to have the same property (ρ^{-1} satisfies $x^3 - x^2 + 2x - 1 = 0$), but most of them fail. The question of obtaining a formula of $\tau(q)$ for the other P.V. numbers is hence still open.

§5. The multifractal formalism.

We first recall some simple facts about concave functions. Let $\tau : \mathbb{R} \to [-\infty, \infty)$ be a upper semi-continuous concave function (it is important to include the value $-\infty$) with *effective domain* Dom $\tau = \{x : -\infty < \tau(x) < \infty\} \neq \emptyset$, and let

$$\tau^*(\alpha) = \inf\{\alpha x - \tau(x) : x \in \mathbb{R}\}$$

be the *concave conjugate* (or the *Legendre transformation*) of τ. It is easy to show that τ^* is also upper semi-continuous and concave, $\tau^{**} = \tau$ and $\tau(x) + \tau^*(\alpha) \ge \alpha x$, for all $x, \alpha \in \mathbb{R}$. (Note that the definition of τ^* is still valid even if τ is not concave, in that case τ^{**} is the concave envelope of τ.) For $x \in$ Dom τ, we let $\partial\tau(x) \subseteq \mathbb{R}$ be the *subdifferential* of τ at x, i.e.,

$$\partial\tau(x) = \{\alpha : \tau(y) \le \tau(x) + \alpha(y - x) \quad \text{for all} \quad y \in \mathbb{R}\}.$$

We will use the following facts frequently [Ro]: $\alpha \in \partial\tau(x)$ if and only if $\tau^*(\alpha) + \tau(x) = \alpha x$, which is also equivalent to $\alpha y - \tau(y)$ achieving its minimum at $y = x$. $\partial\tau^*$ is the inverse of $\partial\tau$ in the sense that $x \in \partial\tau^*(\alpha)$ if and only if $\alpha \in \partial\tau(x)$; (Dom $\tau^*)^\circ = (\alpha_{\min}, \alpha_{\max})$ where $\alpha_{\min} = \inf\{\alpha : \alpha \in \partial\tau(x), x \in$ Dom $\tau\}$ and $\alpha_{\max} = \sup\{\alpha : \alpha \in \partial\tau(x), x \in$ Dom $\tau\}$. τ is said to be *smooth* at $x \in (\text{Dom } \tau)^\circ$ if $\partial\tau(x)$ is a singleton, say $\{\alpha\}$. This means that α is the derivative of τ at x.

τ is said to be *strictly concave* at x if there exists an $\alpha \in \partial\tau(x)$ satisfying

$$\tau(y) < \tau(x) + \alpha(y - x) \quad \text{for all } \ y \neq x. \tag{5.1}$$

For concave functions, the smoothness and strict concavity have a very nice duality relationship, namely

Proposition 5.1. *Suppose τ is a concave function on $\mathbb{R}$.*

(i) *If τ is smooth at $x \in (\text{Dom } \tau)^\circ$ with $\partial\tau(x) = \{\alpha\}$, then τ^* is strictly concave at α, i.e., $\tau^*(\beta) < \tau^*(\alpha) + x(\beta - \alpha)$ for all $\beta \neq \alpha$.*

(ii) *If τ^* is strictly concave at α and the above strict inequality holds for some $x \in \partial\tau^*(\alpha)$, then τ is smooth at x.*

In the rest of this section we assume that $\tau(q)$ is the L^q-spectrum of a positive bounded regular Borel measure as before. Let $\alpha_0 \in \partial\tau(0)$, then for $q \in \partial\tau^*(\alpha)$, $\alpha_{\min} < \alpha \leq \alpha_0$ implies that $q \geq 0$ and $\alpha_0 \leq \alpha < \alpha_{\max}$ implies $q \leq 0$. It is elementary to show that

Proposition 5.2. *Let τ^* be the concave conjugate of τ. Then*

(i) *$(\text{Dom } \tau^*)^\circ = (\alpha_{\min}, \alpha_{\max}) \subseteq (0, \infty)$ and $\tau^* \geq 0$ on $\text{Dom } \tau^*$.*

(ii) *Let $\alpha_0 \in \partial\tau(0)$, then τ^* has a maximum at α_0 with $\tau^*(\alpha_0) = -\tau(0)$. Consequently τ^* is increasing on $[\alpha_{\min}, \alpha_0]$ and is decreasing on $[\alpha_0, \alpha_{\max})$.*

We will first introduce a counting function to reveal the basic relationship between the local dimension of μ and τ^*. Let $\mathcal{B}_h$ denote a disjoint family of closed balls of radii h centered at points in $\text{supp}(\mu)$. For $\alpha_1, \alpha_2 \in (\text{Dom } \tau^*)^\circ$, $\alpha_1 < \alpha_2$, we define the counting functions

$$N_h(\alpha_1, \alpha_2) = \sup_{\mathcal{B}_h} \#\{B : \ B \in \mathcal{B}_h, \ h^{\alpha_2} \leq \mu(B) < h^{\alpha_1}\}.$$

For $\alpha_{\min} < \alpha \leq \alpha_0$, we observe that

$$\#\{B : \ B \in \mathcal{B}_h, \ h^{\alpha_2} \leq \mu(B) < h^{\alpha_1}\} \ h^{q(\alpha+\epsilon)} \leq \sum_{\mathcal{B}_h} \mu(B)^q \leq S_h(q).$$

It follows that, after taking the supremum over all such families,

$$h^{q(\alpha+\epsilon)} N_h(\alpha - \epsilon, \alpha + \epsilon) \leq S_h(q).$$

In view of $\tau(q) = \underline{\lim}_{h\to 0^+} \ln S_h(q)/\ln h$, for any $\xi > 0$, there exists $h_\epsilon > 0$ such that for $0 < h < h_\epsilon$, $S_h(q) \leq h^{\tau(q)-\xi\epsilon}$. Hence for $0 < h < h_\epsilon$,

$$N_h(\alpha - \epsilon, \ \alpha + \epsilon) \leq h^{-q(\alpha+\epsilon)} h^{\tau(q)-\xi\epsilon} = h^{-\tau^*(\alpha)-(\xi+q)\epsilon}$$

and a similar estimate holds for $\alpha_0 < \alpha < \alpha_{\max}$. Consequently we have

Proposition 5.3. *Let μ be a bounded regular Borel measure and let τ be the L^q-spectrum, then be for any $\alpha \in (\mathrm{Dom}\,\tau^*)^\circ$,*

$$\lim_{\epsilon\to 0+} \overline{\lim_{h\to 0+}} \frac{\ln N_h(\alpha-\epsilon,\ \alpha+\epsilon)}{-\ln h} \le \tau^*(\alpha).$$

For the reverse inequality we have

Theorem 5.4. *If in addition τ^* is strictly concave at α, then*

$$\lim_{\epsilon\to 0+} \overline{\lim_{h\to 0+}} \frac{\ln N_h(\alpha-\epsilon,\ \alpha+\epsilon)}{-\ln h} = \tau^*(\alpha).$$

The strict concavity assumption on τ^* at α is roughly equivalent to the smoothness of τ at $q \in \partial\tau^*(\alpha)$ (Proposition 5.2). The advantage of using the strict concavity is that the corresponding inequality is easier to handle. The proof of the theorem is quite elementary and is given in [LN1]. We remark that a similar theorem was proved by Riedi in [Ri1,2] by assuming that the one-sided limit for $\tau(q)$ exists and making use of Ellis' theorem on large deviation. Also another version of the theorem is proved in [F1] by showing that the conjugate of the counting function is τ, and again assuming that the limits in the counting function exist.

Our main goal is to investigate the relationship between $\tau^*(\alpha)$ and the Hausdorff dimension of the set of x such that $\mu(B_h(x)) \approx h^\alpha$ as $h \to 0^+$. More precisely we define

$$K(\alpha) = \{x : \lim_{h\to 0+} \frac{\ln \mu(B_h(x))}{\ln h} = \alpha\},$$

and define $\overline{K}(\alpha)$ and $\underline{K}(\alpha)$ by replacing the $\overline{\lim}$ and $\underline{\lim}$ signs respectively in the above definition. Proposition 5.3 and the Vitali covering theorem yields

Theorem 5.5. *Let μ be a bounded regular Borel measure and let τ be the L^q-spectrum. Let $\alpha \in (\mathrm{Dom}\ \tau^*)^\circ = (\alpha_{\min}, \alpha_{\max})$. Then*
(i) *If $\alpha_{\min} < \alpha < \alpha_0$, then $\dim_{\mathcal{H}} \overline{K}(\alpha) \le \tau^*(\alpha)$;*
(ii) *If $\alpha_0 \le \alpha < \alpha_{\max}$, then $\dim_{\mathcal{H}} \underline{K}(\alpha) \le \tau^*(\alpha)$.*

Our main question is to show that $f(\alpha) := \dim_{\mathcal{H}} K_\alpha = \tau^*(\alpha)$, i.e., the validity of the multifractal formalism. So far a complete answer is still not in sight. The following is a simple case with an elegant proof.

Theorem 5.6. *Let μ be a self-similar measure defined by $\{S_j\}_{j=1}^m$. Let K be the attractor and suppose that $\{S_j(K)\}_{j=1}^m$ are disjoint, then for $\alpha \in (\alpha_{\min}, \alpha_{\max})$, $f(\alpha) = \tau^*(\alpha)$.*

We remark that the disjointness assumption of $\{S_j\}_{j=1}^m$ here is stronger than the open set condition, it is equivalent to the condition in (3.5). Using (3.3) and extending the identity on $\mathbb{R}$, i.e.

$$\sum_{i=1}^{m} a_j^q p_j^{-\tau(q)} = 1, \quad q \in \mathbb{R}, \tag{5.2}$$

we see that $\tau(q)$ is concave, differentiable and the derivative is

$$\alpha(= \alpha(q)) = \tau'(q) = \frac{\sum_{j=1}^m (\ln a_j) a_j^q \rho_j^{-\tau(q)}}{\sum_{j=1}^m (\ln \rho_j) a_j^q \rho_j^{-\tau(q)}}, \qquad q \in \mathbb{R}.$$

Hence, by the digression on concave functions in the beginning of the section, we have $\alpha \in \partial\tau(q)$ and $\tau^*(\alpha) = \alpha q - \tau(q)$. Note that Theorem 3.3 is a special case of Theorem 5.6 when $q = 1$ ($\tau(1) = 0$ and $\tau'(1) = \alpha(1)$). The main part of Theorem 5.6 was proved by Cawley and Mauldin [CM] using $\tau(q)$ as defined in (5.2). That $\tau(q)$ is actually the L^q-spectrum of μ follows from Theorem 3.4 for $q > 0$, and for $q \in \mathbb{R}$ by an independent approach of Riedi [Ri1]. This justifies the multifractal formalism.

In the following we will give an outline of the proof in [CM] which will also motivate the proof of our next theorem in Section 6. In view of Theorem 5.5, we need only show that $\dim_{\mathcal{H}} K_\alpha \geq \tau^*(\alpha)$. The strategy is to redistribute the mass according to (5.2) to obtain a measure ν with support contained in K_α and which has local dimension $\tau^*(\alpha)$ for ν-almost all α. Thus Frostman's lemma will imply $\dim_{\mathcal{H}} K_\alpha \geq \dim_{\mathcal{H}}(\nu) = \tau^*(\alpha)$.

For this we first introduce some more notations. Let $\Omega = \{1, \cdots, m\}^{\mathbb{N}}$ be the product space and let P be the product measure obtained by assigning probability weights $\{a_1, \cdots, a_m\}$ to $\{1, \cdots, m\}$. For $\omega = (j_1, \cdots, j_n, \cdots)$, let $X_n(\omega) = j_n$, be the nth-coordinate projection of Ω onto $\{1, \cdots, m\}$. Also we define $Z_n, Z : \Omega \to \mathbb{R}^d$ by

$$Z_n(\omega) = S_{J_n}(0) \quad \text{and} \quad Z(\omega) = \lim_{n\to\infty} S_{J_n}(0),$$

where $J_n = (j_1, \cdots, j_n)$. Note that the limit exists and is independent of the initial element, which is 0 here. The separation condition in the theorem implies that the map Z is one-to-one from Ω onto K. Let $\widehat{K}_\alpha = Z^{-1}(K_\alpha)$, it follows that

$$\widehat{K}_\alpha = \{\omega \in \Omega : \lim_{n\to\infty} \frac{\ln a_{J_n}}{\ln \rho_{J_n}} = \alpha\}. \tag{5.3}$$

In view of (5.2) we define another product measure Q on Ω using the probability weights $\{a_1^q\rho_1^{-\tau(q)}, \cdots, a_m^q\rho_m^{-\tau(q)}\}$. Let $Y_1,\ Y_2 : (\Omega,\ Q) \to \mathbb{R}$ be defined by $Y_1(\omega) = \ln a_{j_1}$ and $Y_2(\omega) = \ln \rho_{j_1}$. By applying Birkhoff's ergodic theorem to the shift transformation, we have, for Q-almost all $\omega \in \Omega$,

$$\lim_{n\to\infty} \frac{1}{n} \ln a_{J_n} = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \ln a_{j_n} = E(Y_1) = \sum_{j=1}^{m} (\ln a_j) a_j^q \rho_j^{-\tau(q)},$$

$$\lim_{n\to\infty} \frac{1}{n} \ln \rho_{J_n} = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \ln \rho_{j_n} = E(Y_2) = \sum_{j=1}^{m} (\ln \rho_j) a_j^q \rho_j^{-\tau(q)}$$

Taking ratios we have for Q-almost all $\omega \in \Omega$, $\lim\limits_{n\to\infty} \ln a_{J_n}/\ln \rho_{J_n} = \alpha$. This implies that Q is concentrated in $\widehat{K}_\alpha$. If we let $\nu = Q \circ Z^{-1}$, then

$$\lim_{h\to 0+} \ln \mu(B_h(x))/\ln h = \alpha$$

for $\nu-$almost all x in K, so that the support of ν is contained in K_α. Furthermore, the local dimension of ν at x is given by (same reason as (5.3))

$$\lim_{h\to 0} \frac{\ln \nu(B_h(x))}{\ln h} = \lim_{n\to\infty} \frac{\ln a_{J_n}^q \rho_{J_n}^{-\tau(q)}}{\ln \rho_{J_n}} = \alpha q - \tau(q) = \tau^*(q).$$

where $x = Z(\omega)$, $\omega = (j_1, \cdots, j_n, \cdots)$, $J_n = (j_1, \cdots, j_n)$. This is the ν required

For more recent developments, we mention that Theorem 5.6 has been extended to vector-valued self-similar measures constructed by the directed graph method (Edgar and Mauldin [EM] and [Str4]); the finite family of similitudes in the theorem has been replaced by an infinite family by Riedi and Mandelbrot [RM]; Olsen [O1] has refined the definition of $\tau(q)$, using differently sized covering balls instead of the equal size h-balls in (2.1) and developed a parallel theory; and lastly, Arbeiter and Patzschke [AP], Falconer [F2] and Olsen [O3] have obtained analogous results for statistically self-similar measures.

§6. The Weak Separation Property.

In this section we will introduce a new condition on the family of similitudes $\{S_j\}_{j=1}^m$ so as to relax the condition in Theorem 5.6. Let (Ω, P) be the probability space as in the last section. Let $\rho = \min\{\rho_j : \ 1 \le j \le m\}$, we define, for $k \in \mathbb{N}$, the *stopping time* $t_k : \ \Omega \longrightarrow \mathbb{N}$ by assigning each $\omega = (j_1, \cdots, j_i, \cdots) \in \Omega$, the integer $t_k(\omega) = \min\{i : \ \rho_{(j_1,\cdots,j_i)} \le \rho^k\}$. Let

$$\Lambda_k = \{J = (j_1, \cdots, j_{t_k(\omega)}) : \ \omega = (j_1, \cdots, j_i, \cdots) \in \Omega\}.$$

Intuitively Λ_k contains all the indices such that the corresponding contracting ratio are almost equal to ρ^k. Let

$$\Xi_k = Z_{t_k}(\Omega), \qquad \Xi = Z(\Omega)\ (= K),$$

and let μ_{t_k}, μ be the corresponding measures induced on Ξ_k and Ξ respectively; then $\{\mu_{t_k}\}$ converges to μ in distribution.

Definition 6.1. A family of similitudes $\{S_j\}_{j=1}^m$ is said to have the *weak separation property* (WSP) if there exist a $z_0 \in \mathbb{R}^d$ and an $\ell \in \mathbb{N}$ such that for any $z = S_I(z_0)$ (I a finite multi-index), every closed ρ^k-ball contains at most ℓ distinct $S_J(z), J \in \Lambda_k$. (The $S_J(z)$'s can be repeated, i.e., we allow $S_J(z) = S_{J'}(z)$ for $J,\ J' \in \Lambda_k$, $J \neq J'$.)

In view of the fact that the invariant measure is independent of the initial point of the iteration, we will take $z_0 = 0$ for convenience. It is easy to see that $\{S_j\}_{j=1}^m$ will have the WSP if there exists $b > 0$ such that for any $J_1,\ J_2 \in \Lambda_k$, $k \in \mathbb{N}$, and for any $z = S_I(0)$, either

$$S_{J_1}(z) = S_{J_2}(z) \quad \text{or} \quad |S_{J_1}(z) - S_{J_2}(z)| \geq b\rho^k. \tag{6.1}$$

As is known, self-similar measures can be obtained by iterating $\{S_j\}$ starting on any compact set or at any point, the main idea of the WSP is that instead of considering 'set' separation in the iteration, we consider 'point' separation for the iterated points that are distinct. This allows us to include more important cases.

Example 1. Suppose $\{S_j\}_{j=1}^m$ satisfies the open set condition, then it has the WSP. Indeed if we let U be the open set guaranteed by the open set condition, we can fix any $z_0 \in U$ and let $B_r(z_0) \subseteq U$ for some $r > 0$. Let $J_1, J_2 \in \Lambda_k$ with $J_1 \neq J_2$ and let

$$J_1 = (i_1, \cdots, i_n), \qquad J_2 = (j_1, \cdots, j_l).$$

Let n' be the first integer such that $i_l \neq j_l$ and let

$$J'_n = (i_1, \cdots, i'_n), \qquad J'_2 = (j_1, \cdots, j'_n).$$

The open set condition implies that $S_{J'_1}(U) \cap S_{J'_2}(U) = \emptyset$, so that $S_{J_1}(U) \cap S_{J_2}(U) = \emptyset$. Since

$$\rho^{k+1}\text{diam } U \leq \text{diam } S_{J_i}(U) \leq \rho^k \text{diam } U, \qquad i = 1, 2,$$

we have $|S_{J_1}(z_0) - S_{J_2}(z_0)| \geq (2\rho r)\rho^k$. The same holds for $z = S_I(z_0)$ and (6.1) implies that $\{S_j\}_{j=1}^m$ has the WSP.

Example 2. Suppose $1/2 < \rho < 1$ and ρ^{-1} is a P.V. number, then the ICBM μ considered in Section 4 has the WSP. This is in fact a consequence of Proposition 4.3.

Example 3. In wavelet theory, a fundamental equation is the two-scale dilation equation

$$\phi(x) = \sum_{j=0}^{m} c_j \phi(2x - j), \quad x \in \mathbb{R}$$

where $\sum c_j = 2$, $c_j \in \mathbb{R}$. The continuous non-zero L^1-solution has compact support $[0, m]$ [DL1]. Note that if we let $S_j(x) = \frac{1}{2}x + \frac{j}{2}$, $j = 0, \cdots, m$ and $\mu(-\infty, x] = \int_{-\infty}^{x} \phi(x)\, dx$, then μ satisfies $\mu = \sum_{j=0}^{m} \frac{c_j}{2} \mu \circ S_j^{-1}$ as in (1.1). (The coefficients need not be positive here.) If $m > 2$, the family $\{S_j\}_{j=0}^{m}$ does not satisfy the open set condition, but for any $J_1, J_2 \in \Lambda_k$, either

$$S_{J_1}(0) = S_{J_2}(0) \quad \text{or} \quad |S_{J_1}(0) - S_{J_2}(0)| \geq \frac{1}{2^k}.$$

This implies that $\{S_j\}_{j=0}^{m}$ has the WSP. We will return to this class of functions in Section 7.

For $k < k'$, $z_k \in Z_k(\Omega)$, $z_{k'} \in Z_{k'}(\Omega)$, we say that $z_{k'}$ can be reached by z_k if there exists $\omega = (i_1, \cdots, i_k, \cdots, i_{k'}, \cdots) \in \Omega$ such that $z_k = Z_k(\omega)$ and $z_{k'} = Z_{k'}(\omega)$. The following proposition is the main reason to consider the WSP; it allows us to have a good control of the number of paths between the states.

Proposition 6.2. *Suppose $\{S_j\}_{j=1}^{m}$ has the WSP. Then there exists an ℓ_1 such that for $k < k'$ and for $z_{k'} \in \Xi_{k'}$, there are at most ℓ_1 distinct $z_k \in \Xi_k$ that can reach $z_{k'}$*

Proof. Let $S_j x = \rho_j R_j x + b_j$ and let $r_0 = (\max_j |b_j|) \sum_{i=1}^{\infty} (\max \rho_j)^i$. Suppose z_k can reach $z_{k'}$. Then

$$|z_k - z_{k'}| \leq \Big| \sum_{j=k+1}^{k'} \rho_{X_1} \cdots \rho_{X_{j-1}} R_{X_1} \circ \cdots \circ R_{X_{j-1}}(b_{X_j}) \Big| \leq r_0 \rho^k,$$

i.e., $z_k \in B_{r_0 \rho^k}(z_{k'})$. By the WSP, there are at most ℓ distinct z_k in any $(r_0 \rho^k)$-ball if $r_0 \leq 1$, and at most $\ell [2 r_0]^d$ of such z_k if $r_0 > 1$. This proves the proposition.

Let $\partial \tau(0) = [\alpha_0^-, \alpha_0^+]$. Our main theorem is the following:

Theorem 6.3. *Let μ be a self-similar measure defined by $\{S_j\}_{j=1}^m$ and let $\tau(q)$ be the L^q-spectrum. Suppose $\{S_j\}_{j=1}^m$ has the WSP and τ^* is strictly concave at $\alpha \in (\alpha_{\min}, \alpha_0^+)$ (in this case $0 \le q \in \partial\tau^*(\alpha)$), then*

$$f(\alpha) := \dim_{\mathcal{H}}\{z : \lim_{\delta\to 0+} \frac{\ln \mu(B_\delta(z))}{\ln \delta} = \alpha\} = \tau^*(\alpha).$$

In view of Example 1, Theorem 3.4 and the differentiability of $\tau(q)$, we can improve Theorem 5.6 as:

Corollary 6.4. *Suppose $\{S_j\}_{j=1}^m$ satisfies the open set condition. Let μ be a self-similar measure with L^q-spectrum $\tau(q)$. Then $f(\alpha) = \tau^*(\alpha)$ for $\alpha \in (\alpha_{\min}, \alpha_0^+)$.*

Also by Example 2 and Theorem 4.5, we have

Corollary 6.5. *Let $\rho = (\sqrt{5}-1)/2$ and let μ be the corresponding ICBM, then $f(\alpha) = \tau^*(\alpha)$ for $\alpha \in \partial\tau(q), 0 < q \le 3$.*

The proof of Theorem 6.3 is different from Theorem 5.6 in that there is no explicit expression of $\tau(q)$ for μ; furthermore the probability measure P on Ω representing μ does not give tractable information about μ, and we need to look for a new "coding space". For each fixed k, the random variable $Z_{t_k} : \Lambda_k \to \Xi_k$ is given by $Z_{t_k}(J) = S_J(0)$. We will consider the product spaces Ξ_k^n and $\Xi_k^{\mathbb{N}}$ with product measure $(\mu_{t_k})^n$ and $(\mu_{t_k})^{\mathbb{N}}$ respectively. For each multi-index $\mathbf{J} = (J_1, \cdots, J_n), J_i \in \Lambda_k$, we define the truncated index $\widetilde{\mathbf{J}} = (\widetilde{J}_1, \cdots, \widetilde{J}_n)$ where $J_i = (\widetilde{J}_i, J_i')$ and $(\widetilde{J}_1, \cdots, \widetilde{J}_i) \in \Lambda_{ki}$ for each $i = 1, \cdots, n$. (In the case when $\rho_1 = \cdots = \rho_n$, $\widetilde{\mathbf{J}} = \mathbf{J}$.)

Lemma 6.6. *For each k, there exists $g(= g_n) : \Xi_k^n \to \Xi_{kn}$ such that*

(i) *For each $\boldsymbol{\xi} = (\xi_1, \cdots, \xi_n) \in \Xi_k^n$, $g(\boldsymbol{\xi}) = S_{\widetilde{\mathbf{J}}}(0)$ where $\mathbf{J} = (J_1, \cdots, J_n)$ with $S_{J_i}(0) = \xi_i$;*

(ii) *For $\boldsymbol{\xi} = (\xi_1, \cdots, \xi_n, \cdots)$, let $\boldsymbol{\xi}_n = (\xi_1, \cdots, \xi_n)$ and $z_n = g(\boldsymbol{\xi}_n)$, then $\{z_i\}_{i=1}^{n-1}$ is a path that reaches z_n and $\lim_{n\to\infty} g(\boldsymbol{\xi}_n)$ exists;*

(iii) *$g : \Xi_k^n \to \Xi_{kn}$ is at most ℓ_2^n to 1 where ℓ_2 is some fixed integer;*

(iv) *For $z_n \in g(\Xi_k^n)$, $(\mu_{t_k})^n(g^{-1}(z_n)) \le \ell_3^n \mu_{kn}(z_n)$ for some fixed integer ℓ_3.*

By (ii) we can define $g : \Xi_k^{\mathbb{N}} \to \Xi$ such that $g(\boldsymbol{\xi}) = \lim_{n\to\infty} g(\boldsymbol{\xi}_n)$. We use $\Xi_k^{\mathbb{N}}$ as the "coding" space through the map g. We will next construct a measure Q on $\Xi_k^{\mathbb{N}}$ and ν on Ξ as in Theorem 5.6: after strengthening Theorem 5.4 we can find a large k and a subset $E \subseteq \Xi_k$ such that

$$\#E \approx \rho^{k(-\tau^*(\alpha)\pm\eta)} \quad \text{and} \quad \mu_k(\xi) \approx \rho^{k(\alpha\pm\epsilon)}, \quad \xi \in E.$$

Note that all the ξ's in E have "almost" equal probabilities. We define the uniform probability measure on E by assigning the probability $(\#E)^{-1}$ to each

$\xi \in E$ and let Q be the product measure on $\Xi_k^{\mathbb{N}}$. Then Q is concentrated on $E^{\mathbb{N}} \subseteq \Xi_k^{\mathbb{N}}$.

Let ν_n and ν be the induced measures of Q on Ξ_{kn} and $\Xi_k^{\mathbb{N}}$ respectively. We can obtain good control of ν and ν_n using Lemma 6.6, namely, there exists a subset $H \in E^{\mathbb{N}}$ such that for $G = g(H)$, $\nu(G) \geq \frac{1}{2}$, and for $z_n \in G_n = g(H|n), \nu_n(z_n) \approx \rho^{kn(\tau^*(\alpha)\pm\eta')}$.

Hence the scaling exponent of ν at $z \in K_n$ is of order $\tau^*(\alpha) \pm \eta'$. Furthermore, if we let

$$K_\epsilon(\alpha) = \left\{z \in \Xi : \ \alpha - \epsilon < \varliminf_{h\to 0+} \frac{\ln \mu(B_h(z))}{\ln h} \leq \varlimsup_{h\to 0+} \frac{\ln \mu(B_h(z))}{\ln h} < \alpha + \epsilon\right\}, \tag{6.2}$$

then by suitably choosing k we can show that $K_\epsilon(\alpha) \supseteq G$. Frostman's Lemma will imply that the Hausdorff dimension of the set in (6.2) is greater than or equal to $\tau^*(\alpha) - \eta'$, so that $\lim_{\epsilon\to 0+} \dim_{\mathcal{H}} K_\epsilon(\alpha) \geq \dim_{\mathcal{H}} G \geq \tau^*(\alpha)$.

Note that $\bigcap_{\epsilon>0} K_\epsilon(\alpha) = \{z : \ \lim_{h\to 0+} \ln \mu(B_h(z))/\ln h = \alpha\}$. However, this does not imply that $\lim_{\epsilon\to 0} \dim_{\mathcal{H}}(\bigcap_{\epsilon>0} K_\epsilon(\alpha)) = \dim_{\mathcal{H}}(K(\alpha))$. In order to replace the set in (6.2) by $\{z \in g(E^{\mathbb{N}}) : \ \lim_{h\to 0+} \ln \mu(B_h(z))/\ln h = \alpha\}$, we have to replace the fixed ϵ by a sequence $\{\epsilon_k\} \searrow 0$, and the fixed k, Ξ_k^n etc. in the above proof have to be adjusted to a varying k accordingly. The complete proof is given in [LN1].

§7. Scaling functions.

A nonzero function $\phi(x)$ is called a *scaling function* if it satisfies the *two-scale dilation equation*

$$\phi(x) = \sum_{j=0}^{m} c_j \phi(2x - j), \tag{7.1}$$

where $c_j \in \mathbb{R}$. Such functions play significant roles in wavelet theory, constructive approximation theory and fractal geometry. As mentioned in Section 6, Example 3, we can regard the function as the invariance of the IFS with $S_j x = \frac{1}{2}(x + j)$, $j = 0, \cdots, m$. The equation always has a distribution solution [DL1], [Str6], but the main interest is on the existence and regularity of the compactly supported continuous or L^p-solutions (notation L^p_c-solutions); these depend on the coefficients c_j. For the regularity we will use the L^q-Lipschitz exponent defined by

$$\mathrm{Lip}_q(\phi) = \liminf_{h\to 0+} \frac{\ln \|\Delta_h \phi\|_q}{\ln h} \qquad q \geq 1,$$

where $\Delta_h\phi(x) = \phi(x+h) - \phi(x)$. It is easy to show that $\mathrm{Lip}_q(\phi) = \inf\{0 < s : 0 < \limsup_{h\to 0+} h^{-s}\|\Delta_h\phi\|_q\}$. This exponent was introduced by Hardy and Littlewood and is used frequently in harmonic analysis [St]. Its counterpart for measures is the L^q-dimension discussed in the previous sections.

In [DL1] Daubechies and Lagarias proved that if (7.1) has an integrable solution, then $\sum c_j = 2^k$ for some integer $k > 0$. This k is the order of zero of the Fourier transformation $\hat{\phi}$ at 0, and $k = 1$ if $\int \phi(x)dx \neq 0$. We will hence use the natural assumption $\sum c_k = 2$ unless otherwise specified. The existence and regularity of compactly supported continuous scaling functions have been studied in detail by Daubechies and Lagarias [DL1,2] and Colella and Heil ([CH],[H]) using the *joint spectral radius.* The reader can refer to [H] for the very readable exposition.

In the following we will discuss some recent development concerning the L^p-scaling functions. We will use the following linear algebraic set-up (see [DL1], [CH], [MP]): Let $T_0 = [c_{2i-j-1}]_{1\le i,j\le m}$ and
$T_1 = [c_{2i-j}]_{1\le i,j\le m}$, i.e.,

$$T_0 = \begin{bmatrix} c_0 & 0 & 0 & \cdots & 0 \\ c_2 & c_1 & c_0 & \cdots & 0 \\ c_4 & c_3 & c_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & c_{m-1} \end{bmatrix}, \quad T_1 = \begin{bmatrix} c_1 & c_0 & 0 & \cdots & 0 \\ c_3 & c_2 & c_1 & \cdots & 0 \\ c_5 & c_4 & c_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & c_m \end{bmatrix}.$$

For any $g \in L^q(\mathbb{R})$ with support in $[0, m]$, let $\boldsymbol{g}(x)$ be the vector-valued function representing g :

$$\boldsymbol{g}(x) = [g(x), g(x+1), \cdots, g(x+(m-1))]^t, \quad x \in [0,\, 1]$$

and let

$$\boldsymbol{T}\boldsymbol{g}(x) = \begin{cases} T_0\boldsymbol{g}(2x) & \text{if } x \in [0, \frac{1}{2}), \\ T_1\boldsymbol{g}(2x-1) & \text{if } x \in [\frac{1}{2}, 1). \end{cases}$$

It is easy to show that ϕ is a solution of (7.1) if and only if $\boldsymbol{\phi} = \boldsymbol{T\phi}$, equivalently

$$\boldsymbol{\phi} = T_0\ \boldsymbol{\phi} \circ S_0^{-1} + T_1\ \boldsymbol{\phi} \circ S_1^{-1}$$

where $S_i x = \frac{1}{2}(x+i), i = 1, 2$. This is a vector form of the self-similar measure (see Section 3). With no confusion, we use $\|\cdot\|$ to denote the L^q-norm of g and also that of the vector-valued function $\boldsymbol{g}$. For $J = (j_1, \cdots, j_k)$, $j_i = 0$ or 1, we use $I_J = I_{(j_1,\cdots,j_k)} = [a, b)$ to denote the dyadic interval where

$$a = \frac{j_1}{2} + \frac{j_2}{2^2} + \cdots + \frac{j_k}{2^k} \quad \text{and} \quad b = a + \frac{1}{2^k},$$

and g_I means the average $|I|^{-1}\int_I g(x)\,dx$ of g on an interval I. By a 2-eigenvector we mean the right eigenvector of a matrix with eigenvalue 2. We first give a necessary condition for (7.1) to have an L^q_c-solution [LW2].

Proposition 7.1. *Assume $\sum_{j=0}^m c_j = 2$. For $1 \le j < \infty$, let ϕ be an L^q_c-solution of (7.1) and let $\boldsymbol{v} = [\phi_{[0,1]}, \cdots, \phi_{[m-1,m]}]^t$ be the vector defined by the average of ϕ on the m subintervals. Then*

(i) $\boldsymbol{v}$ is a 2-eigenvector of $(T_0 + T_1)$.

(ii) Let $\boldsymbol{\phi}_0(x) = \boldsymbol{v}$, $x \in [0,1)$, and let $\boldsymbol{\phi}_{n+1} = \boldsymbol{T}\boldsymbol{\phi}_n$, $n = 0, 1, \dots$, then $\boldsymbol{\phi}_n(x) = \sum_{|J|=n}(T_J\boldsymbol{v})\chi_{I_J}(x)$, $x \in [0,1)$, and $\boldsymbol{\phi}_n \to \boldsymbol{\phi}$ in $L^q([0,1],\mathbb{R}^m)$.

To look for a criterion for the existence of an L^q_c-solution, the above proposition suggests that we should concentrate on the 2-eigenvector of $T_0 + T_1$. Let $\boldsymbol{v}$ be such a vector, then $(T_0 - I)\boldsymbol{v} = -(T_1 - I)\boldsymbol{v}$. Let $\tilde{\boldsymbol{v}} = (T_0 - I)\boldsymbol{v}$ and let $\{\boldsymbol{\phi}_k\}_{k=0}^\infty$ be defined as in the proposition, then $\boldsymbol{\phi}_n = \boldsymbol{\phi}_0 + \sum_{k=0}^{n-1}(\boldsymbol{\phi}_{k+1} - \boldsymbol{\phi}_k)$ and

$$\begin{aligned}\|\boldsymbol{\phi}_{k+1} - \boldsymbol{\phi}_k\|^q &= \frac{1}{2^{k+1}} \sum_{|J|=k} \Big(\|T_J(T_0 - I)\boldsymbol{v}\|^q + \|T_J(T_1 - I)\boldsymbol{v}\|^q \Big) \\ &= \frac{1}{2^k} \sum_{|J|=k} \|T_J\tilde{\boldsymbol{v}}\|^q \qquad (7.2)\end{aligned}$$

If $2^{-k}\sum_{|J|=k}\|T_J\tilde{\boldsymbol{v}}\|^q \to 0$, it can be shown that it actually converges at a geometric rate, hence $\{\boldsymbol{\phi}_n\}$ converges and the limit gives the L^q_c-solution. Indeed we have the following stronger result [LW2]:

Theorem 7.2. *Suppose $\sum_{j=0}^m c_j = 2$ and $1 \le q < \infty$. Then Equation (7.1) has a nonzero L^q_c-solution if and only if there exists a 2-eigenvector $\boldsymbol{v}$ of $(T_0 + T_1)$ satisfying*

$$\lim_{n\to\infty} \frac{1}{2^n} \sum_{|J|=n} \|T_J\tilde{\boldsymbol{v}}\|^q = 0.$$

Amazingly, under some slightly stronger conditions, the rate of convergence of the above sum actually gives us the L^q-Lipschitz exponent of f [LM].

Theorem 7.3. *Assume $\sum c_{2n} = \sum c_{2n+1} = 1$ and 1 is a simple eigenvalue of T_0 and T_1. Let ϕ be a L^q_c-solution of (7.1), then for $1 \le q < \infty$,*

$$\mathrm{Lip}_q(\phi) = \liminf_{n\to\infty} \frac{\ln(2^{-n}\sum_{|J|=n}\|T_J\tilde{\boldsymbol{v}}\|^q)}{q\ln(2^{-n})}. \qquad (7.3)$$

As an example we consider the dilation equation (7.1) with four coefficients satisfying $c_0 + c_2 = c_1 + c_3 = 1$, $c_0 + c_3 = \frac{1}{2}$. This includes the Daubechies scaling function D_4 where $c_0 = (1+\sqrt{3})/4$, $c_3 = (1-\sqrt{3})/4$. Note that $\boldsymbol{u} = [0,1,-1]^t$, $\boldsymbol{h} = [1,-2,1]^t$ are eigenvectors of T_0 corresponding to the eigenvalues $\frac{1}{2}$ and c_0 respectively; also $T_1\boldsymbol{u} = \frac{1}{2}\boldsymbol{u} + c_0\boldsymbol{h}$ and $T_1\boldsymbol{h} = (\frac{1}{2} - c_0)\boldsymbol{h}$.

By using $\{\boldsymbol{u}, \boldsymbol{h}\}$ as a basis of the subspace $H(\tilde{\boldsymbol{v}})$ spanned by the $T_J\tilde{\boldsymbol{v}}$'s, we can rewrite T_0 and T_1 on $H(\tilde{\boldsymbol{v}})$ as

$$\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & c_0 \end{bmatrix}, \quad \begin{bmatrix} \frac{1}{2} & 0 \\ c_0 & \frac{1}{2} - c_0 \end{bmatrix}.$$

Let $\beta_0 = c_0$ and $\beta_1 = \frac{1}{2} - c_0$. Then the corresponding matrix of T_J on $H(\tilde{\boldsymbol{v}})$ is

$$\begin{bmatrix} 2^{-n} & 0 \\ \lambda_J & \mu_J \end{bmatrix},$$

where $\lambda_J = \beta_0(j_1 2^{-(n-1)} + j_2 2^{-(n-2)}\beta_{j_1} + \cdots + j_n\beta_{j_1}\cdots\beta_{j_{n-1}})$ and $\mu_J = \beta_{j_1}\beta_{j_2}\cdots\beta_{j_n}$, $j_i = 0$ or 1. A direct estimation of the right side of (7.3) yields

$$\operatorname{Lip}_q(\phi) = \min\left\{1, \frac{\ln(2^{-1}(|c_0|^q + |\frac{1}{2} - c_0|^q))}{-q\ln 2}\right\}. \tag{7.4}$$

This formula was also proved by Daubechies and Lagarias [DL3] using a more complicated estimation and the additional assumption that $\frac{1}{2} < c_0 < \frac{3}{4}$.

For the existence and the regularity of L^2_c-solutions, we can adopt the following simpler approach and obtain some sharper results. For $g \in L^2(\mathbb{R})$ with support in $[0, m]$, let $Sg(x) = \sum_{j=0}^m c_j g(2x - j)$, and let $\boldsymbol{a}(g)$ denote the *autocorrelation vector* where each coordinate is given by $a_j(g) = \int g(x + j)\overline{g(x)}\,dx$, $-m \le j \le m$. Let $W = [\omega_{i-2j}]_{-m\le i,j\le m}$ with $\omega_n = \sum_k c_k c_{k-n}$, then a direct calculation yields

$$\boldsymbol{a}(Sg) = \frac{1}{2}W\boldsymbol{a}(g). \tag{7.5}$$

If we use $(Sg - g)$ as an initial function and iterate (7.5) n-times, we have

$$\boldsymbol{a}(S^n(Sg - g)) = \frac{1}{2^n}W^n\boldsymbol{a}(Sg - g).$$

Observe that if $\{S^n g\}_{n=1}^\infty$ converges to a nonzero function in L^2, then the limit will be a solution of (7.1). The left hand side will also converge to 0, and the moduli of the eigenvalues of W restricted to the subspace spanned by $a_n(Sg - g)$ must be less than 2. If we use $g = \sum_{j=1}^m v_j\chi_{[j-1,j)}$, this actually gives us a necessary and sufficient condition for the existence of L^2_c-solutions [LMW]. It yields directly the following simple criterion.

Corollary 7.4. *If $\sum c_{2j} = \sum c_{2j+1} = 1$, and the 2-eigenvalue of W is simple, then (7.1) has an L_c^2-solution.*

By symmetry we can reduce the size of W by half by defining W^+ as

$$w_{ij}^+ = \begin{cases} \omega_{-2j} & \text{if } i = 0, \\ \omega_{i-2j} + \omega_{-i-2j} & \text{if } i > 0, \end{cases} \qquad 0 \le i,\ j \le m.$$

(W^+ is the "folding" of W onto the non-negative coordinates.)

Let $\lambda_{\max}^+ = \max\{\, |\lambda| : \lambda \text{ is an eigenvalue of } W^+ \text{ and } |\lambda| < 2\}$. We have the following sharp regularity estimation of f:

Theorem 7.5. *Assume $\sum c_n = 2$. Let f, α, and $\lambda_{\max}^+$ be defined as above. Let $\alpha = -\ln(\lambda_{\max}^+/2)/(2\ln 2)$ and let k be the largest geometric multiplicity among those eigenvalues λ of W^+ such that $|\lambda| = \lambda_{\max}^+$. Then*

$$\frac{1}{h^{2\alpha}|\ln h|^{k-1}} \int_{-\infty}^{\infty} |\Delta_h f(x)|^2\, dx = p(h) + o(1) \quad \text{as } h \to 0^+,$$

where p is a nonzero, bounded continuous function and $p(2h) = p(h)$. In particular $\mathrm{Lip}_2(\phi) = \alpha$.

The theorem is similar to Theorem 4.4 where W^+ and $\{0, \cdots, m\}$ here correspond to the T^+ and Γ_0^+ there. The basic idea of the proof is also similar, but there are additional complications because the coefficients are not positive [LMW]. We remark that Corollary 7.4 had also been proved by Cohen and Daubechies [CD] and Villemoes [V] through a Fourier transformation approach. They also proved the last statement of Theorem 7.5 in terms of the Sobolev exponent and Besov spaces and under the additional condition that $\sum c_{2j} = \sum c_{2j+1} = 1$.

There are also physical models on the multifractal structure of functions constructed from the cascade algorithm (e.g., Frish and Parisi [FP] investigated the Hausdorff dimension of the set of points of Lipschitz order α in the velocity field of a turbulence). For a compactly supported continuous ϕ, we define for $q > 0$,

$$\tau(q) = \liminf_{h \to 0^+} \frac{\ln \int |\Delta_h \phi(x)|^q dx}{\ln h} - d.$$

It follows that for $q \ge 1$, $\tau(q) = q\ \mathrm{Lip}_q(\phi) + d$. The corresponding multifractal formalism is

$$\tau^*(\alpha) = \dim_{\mathcal{H}}\{x : \lim_{h \to 0^+} \frac{\ln |\Delta_h \phi(x)|}{\ln h} = \alpha\}.$$

In [J] Jaffard had carried out a detailed study of the multifractal formalism for general functions, and for self-similar functions assuming the open set condition. Note that the class of scaling functions is not included in his study because the corresponding IFS generate overlaps. In [DL3] Daubechies and Lagarias made a first attempt to understand the formalism for the scaling functions. They considered a small class of such functions (see the example following (7.3)) and calculated the L^q-spectrum and the local dimension spectrum $f(\alpha) = \dim_{\mathcal{H}}\{x : \lim_{h\to 0^+} \ln|\Delta_h\phi(x)|/\ln h = \alpha\}$ explicitly. It was found that the L^q-spectrum has one non-differentiable point (it shows in (7.4)) and that $f(\alpha)$ has a jump at $\alpha = 1$. The multifractal formalism is not true in their case. However, $\tau^*(q)$ still equals the concave envelope of $f(\alpha)$.

The calculation used in [DL3] is very restrictive and there is no general theory in this direction yet. It is not known whether the point $\alpha = 1$ for which $f(\alpha)$ fails the formalism is genuine or special (it is known in harmonic analysis that the Lipschitz exponents at the integer point behave slightly differently [St]). Since the IFS for the scaling function satisfies the WSP in section 6, and that the results from [DL3] still agree with Theorem 6.3, it will be interesting to extend the proof of the theorem to cover the case of signed weights for measures and functions. Moreover Theorem 7.3 provides a general expression for the Lipschitz exponent of the scaling functions which will also be useful to determine their multifractal structure.

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